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THE MATHEMATICS TEACHER

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THE PROJECT METHOD AND THE SOCIALIZED RECITATION

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INTRODUCTION

In this paper we propose to consider the Project Method and the Socialized Recitation from the point of view of their contribution to pedagogy in general and to the teaching of mathematics in particular. It will be necessary, naturally, to define these terms, and in defining them we shall find clues to the aspects of them that constitute their distinctive and valid contributions as well as to those aspects that we consider their shortcomings.

The Project Method and the Socialized Recitation may be defined as (1) a critique of current practice; (2) a psychology of learning; (3) a way or method of teaching. In all these respects the Project Method and the Socialized Recitation have made significant contributions to the thinking of teachers and educators.

I. THE PROJECT METHOD AND THE SOCIALIZED RECITATION AS A CRITIQUE OF PRACTICE

Current practice in teaching is still largely formal and therefore lacking in vitality and appeal to the child. Subject matter receives emphasis, logical organization of it is the aim, and the adult world, of which the teacher is the spokesman, is the dominant consideration. As against these, the proponents of the project method insist on the fact that subject matter as such does not exist for the child and that logical arrangement is an adult interest and an acquired taste.

The changes which the teaching of reading has undergone are

apt illustrations of the changes which can be predicted for teaching in other fields. Logical consideration at one time pointed simply, clearly, and unequivocally to the fact that the child should be taught his alphabet first, that he should then be taught how to combine the letters into words, and that he should later be taught to combine words into sentences. Psychologically, of course, this was not the best method. For, first comes the sentence, nay the story. Prevailing methods in the teaching of reading have already incorporated this important consideration.

In most other work, particularly in high school work, the material is still subject matter, organized on a logical basis by experts in the subject matter. Textbooks in algebra are still in use which exhaust all possible cases in removal of signs of aggregation before proceeding to problems which involve simple parentheses; they give exhaustive (and exhausting) work in "apartment house" fractions before they proceed to the solution of equations involving fractions, and so on.

Only in very recent years has any thought been given to questions like the following: When is the child psychologically prepared to learn what the negative number is? What is the proper place, psychologically speaking, for introducing graphic work? When is the child ready to make some of the simpler geometric constructions?

We shall take up later the question of the extent to which secondary school teaching may begin reasonably to encourage and expect logical organization of subject matter.

What is the result of forgetfulness of pedagogues concerning the nature of the child and of their predilection for formal organization and presentation of material? The child is not enlisted in the task. The child does not identify himself with the school task. The teacher is the active partner; the child is a passive and indifferent observer of the process. An interesting indication of this fact is the way in which both school children and teachers refer to the extra curricular elements of school life as the activities. By implication, the school tasks are the passivities.

This detachment of the child from his school tasks, the lip service which he often learns to give—by way of giving the teacher his due—has a bad pedagogic consequence which we shall consider at greater length in the next section. Here we wish

merely to refer to the dualism in the child life to which it leads —and to the disintegration of personality in which it sometimes results.

The child must see value in the task unless he has faith in the adult world and its representatives and in the deferred values which are inherent in the task. Often this faith is lacking, and often, too, the values are too long deferred. Then the school finds it necessary to resort to marks, promotion, and the other forms of currency. These the child has been known occasionally to circumvent or he has been as indifferent to them as to the tasks themselves. And sometimes the deferred values simply do not exist, to the extent which is claimed for the tasks. Teachers of mathematics will find interesting in this connection Chapter II of Thorndike's *Psychology of Algebra,* which shows in what small measure the utilitarian values claimed for much of the work in algebra are realized.

Finally we mention the fact that school work fails to furnish the basis that it should furnish for social coherence of the group in which the child finds himself. The school recitation, as we know it, is one in which the relation is a *many-one* relation between pupils and teacher, one in which the many are an aggregation of individuals, not a social organism. The child recites, the teacher approves in greater or less degree, the mark being the index of the degree of approval. The pupil who is not "reciting" is interested only with reference to the chances of his being the next one called to the firing line.

In defining the Project Method and the Socialized Recitation as a critique of current practice we have seen that proponents of them are critical of current practice because it is formal and unpsychological, because it leads to unfortunate behaviors and attitudes in children, and because the school studies do not furnish the pupil with the needed basis for social coherence and the needed motive for identifying himself with the group.

In short, the pupil is not inspired or prompted to identify himself with the school task with which he is faced or with the social group in which he finds himself.

II. The Project Method and the Socialized Recitation as a Psychology of Learning

Real learning is achieved only when it is done with all one's heart and with all one's soul—also with all one's body. Learn-

ing is efficient only to the extent to which these conditions are met. The entire child, not just part of him, must be enlisted. It is now generally known that merely imparting information is not teaching and that mere receiving of it is not learning. There must be favorable learning conditions which will bring about a mind-set in the pupil, an active attitude in which his whole being is enlisted. The act must have *purpose* which is the pupil's purpose; the *end* must be one which the pupil will regard as worth achieving; the *plan* must be one which the pupil regards as effective. The group with which the pupil works, if he works in a group, must be an integrated group of which he is a willing and cooperative member and of which the teacher likewise is a member. Under these conditions, the pupil identifies himself in his group and with his task.

The mind-set thus achieved spells readiness, leads and inspires to persistent action, and enhances the chances of success. The resulting satisfaction produces deep-seated modification in behavior. This is true learning.

The school teacher's task is to find problems that are gripping and that will stimulate to purposeful activity. Learning through purposeful activity has as important by-products what Professor Kilpatrick calls associate learning and concomitant learning. By the one he means certain incidental skills, abilities and bits of information that logically belong in "subject" categories, and that should be so organized later. By the other he means attitudes towards life's tasks—attitudes that should be characterized by a spirit of adventure, a readiness to face situations that are presented by an ever-changing world.

III. The Project Method and the Socialized Recitation as Ways of Teaching

The project method in the light of the psychology of learning to which it subscribes, aims to introduce favorable learning conditions, by which are meant conditions under which the pupil will easily and naturally identify himself with the task. Such conditions are conditions that most closely resemble life situations as the child sees life.

Here we refer, in passing, to the possibility of a contradiction in terms and therefore differences among school and teachers in the interpretation of the project method, depending on which of

the two contradictory terms is the more clearly held in view. Is the child life the real life or are the adult ways the real life which the schools are to simulate? Some, seeing in the schools only a means of preparing the child for worthy adult life, would have the school a miniature adult world in which the children, practicing in the ways of the adult world and exercising a certain amount of self-direction in play and in work, will become prepared for real adulthood in a real, adult world. Others, on the contrary, insist that worthy child life makes for worthy adult living. It is the child's interests in the world of childhood and active self-direction in things that appeal to children without reference, in the early stages, at least, to deferred values and purposes, that will tend to give the child a sense of significance, a sense of personal worth, and a feeling of real worthwhileness of it all—and thereby an integration of aim and purpose that makes him a person.

Both interpretations have shown results that are gratifying to their proponents. In schools which essayed either method, the reports indicate, children have found the school real and school life worth living. In both, the children have identified themselves with the school tasks.

There are types of projects the character of which it is important to realize in order to avoid dangers to which we shall refer at length later. Professor Kilpatrick distinguishes the following types of purposeful activity:

1. *Producer's Project.*—In mathematics one instance of producer's project would be the learning of square root (as reported by a school) by a pupil who wanted to learn how to compute the length of beam needed for a sloping roof of a shack which he was constructing.

2. *The Consumer's Project.*—Here the purpose is not beyond the present moment. Some object is to be produced for its own sake. Such was the work of *Shanks* when he computed the value of π to 707 places. Such might be the project of a child who is interested in making magic squares. Such also is that of a little girl of ten who wanted problems given to her because she likes to see how numbers "behave." (The significant word "behave" is hers.)

3. *The Problem Project.*—Here the individual enters upon the solution of an intellectual difficulty—one which may or may not

have application beyond the present, but which unsolved is for the individual a refined irritant. For Gauss the so-called imaginary number was such an irritant. For the child in the junior high school the problem of summarizing group characteristics can be such an irritant and stimulus to activity, to the extent that he may want to learn about the statistical methods which have been evolved for the purpose.

4. *Finally, the Skill Project.*—Because of some other end which the individual wishes to realize, he may be willing to subject himself to a discipline whereby he will acquire a skill which is a *means* toward realizing that end. Quicker and more accurate ways of adding, not interesting in themselves, are made worthwhile as a means of satisfying work in statistics. Changing the subject of a formula can be interesting on its own account, and comes under the projects referred to as problem project. But skill in the solving of equations is necessary to this end though it often fails to enlist pupil activity. However, when a reasonable standard of skill is suggested by the teacher to the pupil (reasonable from the point of view of the pupil's age and capacity) the child is in a position to know the limits to which he may expect to go and has a means of measuring his own progress. The recently devised practice tests in all fields serve this purpose. The exercises devised by Schorling, Clark and Lindell, and by Smith, Reeve, and Morss are examples of them in the field of mathematics.

We see that the Project Method demands of the teacher that he organize his material shrewdly into gripping problem material, and that the four types of project, if the division be valid, make it possible for the teacher to bring in much material which would be excluded if he limited himself to the narrow interpretation of the real and the concrete. For the teacher of mathematics the suggestion is that mathematics be regarded increasingly as a way of dealing with reality, since number and necessary conclusion are aspects of reality in the phenomenal world and in the world of thought.

The limitations of this article make it impossible for us to deal at this time with Professor Kilpatrick's notion that arithmetic and other abilities or knowledges are pieces strewn along the way of experience as obtained through the project. Professors Bagley and Thorndike have pointed out the dangers of a naïve

reliance upon such piece-meal and patch-quilt learning of the arithmetic or of developing any ability.

The Socialized Recitation, as we have seen, has as its object the self-identification by the pupil with his social group. We wish to consider briefly the need for such self-identification. From the point of view of a better schooling it is necessary. Says Professor Dewey:[1] "I should like at this point to refer to the recitation. We all know what it has been—a place where the child shows off to the teacher and the other children the amount of information he has succeeded in assimilating from the textbook. From this other standpoint the recitation becomes preeminently a social meeting-place; it is to the school what spontaneous conversation is at home, except that it is more organized, following definite lines. The recitation becomes the social clearing house, where experience and ideas are exchanged, and subjected to criticism, where misconceptions are corrected, and new lines of thought and inquiry are set up."

The need for a change in the school is but an index of the larger need. It is becoming increasingly necessary for the individual to be at home in groups with the members of which he has little of the superficial aspects in common. Populations shift more easily, so that the adaptation of the individual must not be in a too narrowly circumscribed area or too small a group. The annihilation of time and space as achieved by advance in technology has given the individual a greater and more varied number of neighbors at his very door. For these reasons more and faster social bonds are necessary, and the capacities for creating them must be realized wherever and whenever these capacities can be exercised. The school house is both an opportunity and the occasion of a duty in this direction.

In the socialized recitation the pupil is not addressing his remarks to an inquisitionally minded teacher. He is addressing his remarks to the group of which both he and the teacher are members. The criticism of the pupil's presentation comes from the members of the group, of whom the teacher is but one, although in truth and, properly, a very (and often the most) influential member. In fact, the pupil does not recite in the traditional question-answer method, the child is an expositor. His reward is

[1] Dewey, John, Twenty-sixth Yearbook of the National Society for the Study of Education, Part II, p. 177.

the exercise in giving an exposition and the correction or approbation which his mates vouchsafe the exposition.

The problems themselves are often the outgrowth of a large project or purpose which has its origin in the group. The purpose may thus be a collective one. The carrying out of the project is therefore social, although, as is often the case, aspects of it are dealt with by sub-groups or individual members of the group. Purposes and experiences are shared.

Through such collective endeavor the individual grows along one of the most precious and most needed of directions. He acquires knowledge or skills which are, of course, important. More important, however, is the *manner* in which he acquires knowledge and skill, for it is the *ways* of learning which will be the permanent possession.

The child who has lived the school life in which purposive learning is the dominant activity will continue to learn even in the absence of the traditional school agencies and agents; the child who has shared experiences with his fellows in the realm of school activities will find such sharing normal in his adulthood. The counting house ethics, which dominates the school room in which the child purchases the good will of the teacher through recitations that are satisfactory to the teacher, must be replaced by a social ethics so that the peculiar abilities of each member may be realized and valued as precious contributions to a collective endeavor.

IV. Critical Estimate of the Project Method and the Socialized Recitation

The emphasis on the need for purpose in learning and of a socialized milieu for that activity is no small contribution to pedagogy. It is, on the contrary, of vast significance. It brings home once more to the teacher the truth that the child is the starting point of the teaching process and the realization of his capacity for growth is the end or aim. Systems and individuals that give thought to the question which is thus raised cannot but be vitalized by coming thus at grips with the basic problem in educating.

In what we shall now say concerning the project method and the socialized recitation we ask the reader to understand that it is in the interest of a clearer comprehension of what we deem

to be their essentially valid contributions that we call attention to some of their limitations.

1. We must not interpret purpose in too narrow a sense. In his "How We Think," Professor Dewey tells us: "It does not pay to tether one's thoughts to the post of utility by too short a rope." Purpose must not be too narrowly utilitarian. We must not insist that our children wear blinders in their way through the school world.

Children have intellectual interests. We must bear in mind the four types of projects which Professor Kilpatrick distinguishes for us. Schools and teachers have been known to see only some tangible, palpable construction such as a house or a dress or a store as a project. Children have intellectual interests, and though this perversion may be the consequence of association with unregenerate adults—they sometimes do *want* to learn in order to know. Such learning has high retentive value, as Professor Bagley points out.

2. In attempting to simulate "reality" in the school world we are in danger of making too acceptable to the growing generation much that unhappily is but that ought perhaps not to be. There is danger that the school may thus become an instrument for perpetuating the status quo, whereas it would be more truly descriptive of the purpose of the school to make it an agency for a progressively changing society. Through the schools society should attain higher and better levels of living.

3. The end in view sometimes undergoes changes in character while we are in the act of working to attain it. Sometimes it changes merely in that it becomes more clearly defined in terms of material and limitations. Sometimes it changes entirely in character as the material and the activity give intimations of an end more comprehensive and more desirable. Sometimes the end attained proves—and it is well that it prove—to be but a step toward a goal beyond.

4. It is not always possible and it is not even always desirable that the purpose originate with the children. On this score, Professor Dewey [1] has this to say: "There is a present tendency in so-called advanced schools of educational thought to say, in effect, let us surround pupils with certain materials, tools, appliances, etc., and then let pupils respond to these things according

[1] Dewey, John, *loc. cit.*, p. 173 et seq.

to their own desires. Above all, let us not suggest to them what they shall do, for that is an unwarranted trespass upon their sacred, intellectual individuality, since the essence of individuality is to set up ends and aims.

"Now, such a method is really stupid. For it attempts the impossible, which is always stupid; and it misconceives the conditions of independent thinking. There are multitudes of ways of reacting to surrounding conditions, and without some guidance from experience these reactions are almost sure to be casual, sporadic and ultimately fatiguing, accompanied by nervous strain. . . . Moreover, the theory literally applied would be obliged to banish all artificial materials, tools, and appliances. Being the product of the skill, thoughts, and matured experience of others, they would also, by the theory, 'interfere' with personal freedom.

". . . There is no spontaneous germination in the mental life. . . . If the teacher is really a teacher, and not just a master or 'authority,' he should know enough about his pupils, their needs, experiences, degrees of skill and knowledge, etc., to be able (not to dictate aims and plans) to share in a discussion of what is to be done and to be as free to make suggestions as any one else."

5. Some topics and perhaps some subjects do not lend themselves to treatment in a Socialized Recitation as that term is usually understood. Particularly is this likely to be true in mathematics. Only a rich background makes it possible for an individual or a group to introduce to another individual or group the concepts in their mathematically accepted sense. A subject in which the tools themselves are the object of study—as distinguished from one like literature or geography or history, where the tools are the ability to read and the books which contain the material—is necessarily one that must be introduced carefully and precisely at the very beginning. Errors in the initial steps become hopelessly cumulative in their effect later. An instance of what we mean is furnished by the narrow and misleading use of terms like *fraction, invert, transpose, cancel.* Only the teacher who has given thought to experience in this connection can anticipate the possible sources of error and consequent mischief, and can lead to avoidance of them by carefully designed practice in the correct usages. These "correct" usages are often the outgrowth of centuries of race experience and experimenta-

tion. We must not expect the child to repeat in his short span of school life the entire history of the race in all its details.

6. There is the danger, already manifest in much of the literature of the "progressive" or "new" education, that unreasoning taboos and conditions of orthodoxy may be established. Of these taboos a significant one is against subject matter as such. So fearful are the "new" educators of the abuses to which single-minded devotion of schools to logical organization has led in the practice of teaching, that they would discard logical organization of material altogether. It is not necessary, it is not reasonable to go all the way in this rejection of systematization of knowledge. With children of elementary school age it is well to avoid abstractions which threaten to become verbal substitutes for experiences which they should first acquire. But children of junior and senior high school age need some organization of their experiences. These schemes of organization may indeed be different from the schemes that are used by research workers in the universities. We now speak of general science instead of speaking of biology, or botany, or chemistry in the work of the junior high schools. Similarly, general mathematics is the name of the mathematical material of some of the junior high school grades. Of the organization of material there is need. Not only that, but the assembling and the organizing of the material on some logical basis is a worthwhile educational experience for children of high school age.

V. Conclusion

We do not wish in this study to conclude with a note of adverse criticism. We prefer to stress the contribution which these methods have made in school procedure. Through daily living on a high plane, through daily living of an integrated life, through daily realization of significance in tasks assumed and carried through to meaningful conclusions, the child is to be prepared for adult life on a high level of creative endeavor and courageous attitude toward an ever moving and never arriving world. By responding to the best that these methods offer, the teacher is rejuvenated through a keener sensitiveness to the problems with which his pupils are wrestling, and is ennobled through the effort to evoke such fineness as is latent in the young people with whom he is associated—no mean reward, and one which is the teacher's unique compensation.

THE FORMULA IN SECONDARY EDUCATION

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We all know that due to external pressure the Mathematical Association of America appointed a committee called the National Committee on Mathematical Requirements and that this committee made recommendations for the reorganization of mathematics in secondary education. Through external pressure, I say, this was brought about; for either mathematicians were too near to be able to focus on the large defects that appeared to outsiders or were too busy amplifying and extending the knowledge brought down from the ancients to realize that times were changing and that if we were to give every child a high school education, we must use a different procedure from that of the old world where only a select class receive a broad education.

With the report of the National Committee there came the greatest upheaval in the field of algebra. This consisted in deleting the technique and in changing the arrangement and presentation of the remaining facts. Some authors made the equation the basis for all work in algebra, others the graph, and still others used the formula as the foundation for the whole structure. It is with the last that I wish to thrust in my humble oar.

Mathematical truths always have two sides, "with the one they face and have contact with the realities of the world. With the other they face and have relations with one another." The history of mathematics is a tale of ever-widening development on both of these sides. There have always been eras of demand for the distinctly and purely practical and eras in which the theoretical was uppermost. These are exemplified by the Egyptians for the former and by the Greeks for the latter. But these different currents of progress must not be thought of as independent streams. They are rather two currents running side by side in the same stream some times meandering a little

way apart, 'tis true, but always combining to form the mighty rapids.

Thorndike, in his "Psychology of Algebra," says: "The main service of algebra, as the psychologist sees it, is to teach pupils that we can frame general rules for operating so as to secure the answer to any problem of a certain sort, and express these rules with admirable brevity and clearness by literal symbolism."

In accordance with this view, algebra should be introduced to the young pupil as a symbolic language specially adapted for making concise statements of a numerical kind about matters with which he is already more or less familiar. Earliest lessons should teach the use of the formula, illustrations being drawn from the "Engineer's Handbook," etc. A little later comes manipulation to find truths not perceived before. Pupils should be made to feel that formulas and manipulation always refer to realities beyond themselves and that we are not manipulating formulas just to show that we can do so. It is not necessary to say that incompetence in mathematics and distaste for it nearly always comes from the neglect of this fundamental teaching principle. Even in the case of those who have a natural fondness for the technique of mathematics, the same neglect often leads to an astonishing blindness to the real significance of mathematical ideas and operations.

T. Percy Nunn, the great English mathematician, has this to say: "Algebra regarded as generalized arithmetic should have no formal beginning. As soon as the child, who sees the teacher write upon the blackboard: area = length times width, can translate this into the words: "To find the area of the floor I must multiply its length by its breadth," he has without knowing it, already begun his study of the subject. What the teacher has set before him has the two characteristics of a formula: (1) it is a statement of a general rule applicable to any one of a definite class of problems and (2) the statement is expressed in a conventional form chosen for its properties of conciseness and ready comprehensibility. By his twelfth year lessons in arithmetic and science should have afforded the pupil abundant opportunity of learning to write down and use simple formulas of this kind."

It can not be denied that formulas have these distinct advantages over the written rule. (1) Words and phrases as the vehicles of ideas are replaced by symbols with a consequent gain

in clearness and conciseness. (2) A formula, consisting of an arrangement of symbols, is free from the ambiguity which often besets the arrangement of verbal units into a sentence, and is besides, a more effective vehicle of a complicated meaning.

Formulas must of course never be used unless the pupil clearly understands the processes which they prescribe. In other words he is entitled to use a formula only if it represents genuine results of his own thinking. He may then with advantage write it at the head of his calculations as a memorandum of the process which he intends to employ. Used in this way, the formula makes for greater clearness both of the pupil's thinking and of his written statements.

The use of words in an abbreviated form, supplies a natural transition to the stage where we replace the phrases of a verbal formula by single letters. The use of single letters can best be explained by teaching the class to regard formulas as shorthand memoranda of the rules which they have already established in the course of their work and are constantly needing. The principles of this shorthand are to represent such constantly recurring words as multiply, divide, square, and the like by conventional symbols and to reduce other words or verbal expressions in the full statement of the rule to single letters, chosen so as to suggest those words or expressions as readily as possible to the reader of the memorandum. Reducing Area to A, rate of interest per annum to r and so on. When the verbal statement of the rule contains a numerical constant, the practice of placing it before the literal symbols must be taught. Thus, the rule for area of a triangle is, in accordance with this convention, to be written neither in the form $A = a\frac{1}{2}b$ nor $A = ab\frac{1}{2}$, but in the form $A = \frac{1}{2}ab$. We can say that it really means the same but seems more awkward to a person used to writing it the other way.

Whenever the subject of formulas is approached, the method should be somewhat as follows, say for the area of a rectangle: If necessary we should develop the fact that the area is equal to the number representing the length multiplied by that representing the width because one is the number of square units in a row and the other the number of rows. Then we may say to the class, "We may now write the rule on the blackboard but it is unnecessary to write every word in full; for you will have

no difficulty in knowing what I mean if I shorten it down to the following: area of rect. = length $\times$ width." Also to say to them: "There are a great many persons who have constantly to make use of notes of this kind. They are such people as engineers, who have to keep notes of all sorts of rules in regard to weights which their materials will bear, etc.; electricians, architects, sailors, etc. Some of their rules are so complicated that their notes would be very cumbersome even if they shortened the words down to vol. for volume and ht. for height, etc., and employed symbols such as = (equality sign) and $\times$ (times). They therefore find it necessary to use a kind of shorthand in which they can express their memoranda much more briefly even than that we have expressed the rule for finding the area of a rectangle." Someone in the class will suggest using "A" for area of rectangle and l and w for length and width respectively. The rule is now $A = l \times w$, but if we make up our minds never to use more than one letter to represent a word or group of words, the formula may be shorter still. We can agree to indicate that two numbers are to be multiplied together simply by writing the letters which are the shorthand descriptions of them side by side. Upon this plan our formula becomes $A = lw$. Such a formula is, remember, a shorthand way of writing down the sentence, "The area of a rectangle is obtained by multiplying the length by the width." "Is obtained by" is equivalent to "equals" while multiply is implied by the fact that the letters are written side by side. If it is necessary to make clear in your notebooks what words the various letters stand for, it is best to write as follows: A = area of rectangle, etc.

At this point a great deal of practice should be given to expressing in shorthand, a great many rules of arithmetic, mensuration and elementary science. Perhaps the method I have described with the rectangle may be abridged or not necessary at all with some particular group but there should be some introduction of the sort and generally it is necessary to take some time even in the ninth grade to discuss the difference between area of a rectangle and its perimeter.

Notice that nothing is said of an equation but that the formula is merely a shorthand transcription of a verbal rule or other statement. At this time it is hardly possible to lay too much stress upon the importance of cultivating a neat and orderly

way of setting down the steps in an algebraic argument. A piece of algebraic symbolism should be as capable of straightforward and continuous reading as a passage from a newspaper. To achieve this end the teacher will find it a sound rule never to permit a line to contain more than two expressions connected by the sign of equality and to insist upon the pupil's setting the signs of equality, in successive lines of the argument, directly underneath one another. We often see statements such as this: The pupil is to multiply 32 by 56 and divide by 9 and he says, $32 \times 56 \div 9 = 32 \times 56 = 1792 \div 9 = 199\ 1/9$.

There is a great deal of discussion as to whether the formula or the graph should be taught first. It is obvious that the graph shares many of the properties of the formula. Like the formula it can be used to bring out and express the law which underlies a number of concrete numerical facts. Like the formula it delivers its message in a form readily taken in by the eye; but it is inferior to the symbolic formula in many respects. Its accuracy depends largely upon mechanical or non-intellectual conditions such as the skill of the draftsman and the exactness of the squared paper. It is less compact and less easily reproduced. Its message is frequently inarticulate and obscure. For these and similar reasons it should be regarded as a subsidiary algebraic instrument which does its duty best when it either leads up to a formula by which it may itself be superseded, or serves to unfold more fully the implicatons of the formula whose properties have been only partially explored.

But graphic methods were employed as an effective instrument of mathematical thinking before algebraic symbolism had been developed beyond the rudiments. Thus, the Greeks who never succeeded in producing a satisfactory algebraic method, yet performed analytic feats of high importance with the aid of graphic forms. The superior vividness of graphic modes of expression suggests the conclusion that the young pupil should be taught their simpler uses before he makes acquaintance with the harder though more powerful instrument, the formula. We all know that a child will express himself in pictures long before he has learned to command the more abstract medium of written words. My point is this then, that in seventh or eighth grade or in both, some graphic work should be given as of simple interest, cost of articles at a given price each etc., then at the be-

ginning of the ninth year, the formula should be taken up in connection with the graph.

In connection with graphic representation, I think we should always encourage the child to draw a picture when practicable. If he draws a rectangle, he can scarcely fail to realize what the perimeter is equal to, if he knows what that word means.

Formal work in algebra—as distinguished from the incidental use of symbolism in arithmetic and elementary science—is here planned to begin with lessons intended to cultivate the formula as an instrument of mathematical statement and investigation. When it is considered how essential is their use in a vast range of trades and professions—from plumbing to dreadnaught building—it is hardly extravagant to say that facility in the working, interpretation, and application of formulas is one of the most important objects at which early mathematical studies can aim. A beginning at this point therefore, secures the tactical advantage of giving the pupil his first view of the subject on its most obviously useful side.

The cultivation of the formula involves four distinct elements: (1) practice in analyzing arithmetical processes and rules of procedure; (2) practice in symbolizing the results of analysis; (3) practice in interpreting given pieces of symbolism; (4) practice in substitution. The first two constitute the art of formulation; the second two the art of using formulas. It is advantageous to give separate study to these two sides of the work.

In some examples it is well to give a numerical example before demanding a formula. This is of course intended to help the pupil to formulate the essential rule. In other cases where he can not formulate the rule without such an instance, he should invent one for himself. The numerical example should not be worked out but should be set down in such a way that all the steps involved in obtaining the result are clearly exhibited. As a rule these steps will be taken one by one without any clear consciousness of the plan of procedure as a whole. The pupil should then analyze the work so as to make himself clearly cognizant of the details of the general plan, apart from the particular numbers in which, in this case, the plan is realized. The formula is, as we have already seen, nothing more than a statement of the plan in accordance with a conventional system

of symbolism. The formula obtained by analysis is of course not a description of the special case. The special case is used merely as a challenge to the pupil to formulate the procedure which he would be bound to adopt in any case. As he gains experience it should become less necessary to begin with a numerical example. At each step in the evolution of the formula, he should determine what he would do next if he were concerned with a particular case, and should write down, symbol by symbol, his statement of procedure without needing to have the special case as a whole before him. Finally he should cease to be conscious that he is making any appeal at all to numerical instances and should handle his symbols exactly as if they were figures.

This principle is of quite fundamental importance, viz. that in the algebra lesson the pupil shall always feel that he is face to face with something in the real external world and that his business is to give in symbolism an account of its behavior in its numerical aspect. The teacher should keep this prominently before him at every stage. Thus, whenever the facts underlying an example are unfamiliar, something should be done to make them more real to the class before the analysis is attempted. At this stage one should of course avoid cases where it would be necessary to make use of signed numbers.

It is well to have some formulas translated into verbal terms, being careful to have a ready and exact rendering. But a greater need is to cultivate the art of substitution. The meaning of substitute is clear to all children by referring to the substitute on a football team. Here is where the care should be shown in setting down results as mentioned before. Formulas should be chosen as much as possible for the interest of their subject-matter as well as for their value as exercises in substitution. This is not difficult, for the symbolic forms which have most frequent practical use naturally offer the most profitable field for exercise in substitution. Many examples in substitution can be obtained from Mechanical, Civil or Electrical Engineer's hand books. In some cases the formulas admit of simple practical applications. A few such applications will do more to illustrate the value of algebra than many formal lessons. The teacher should, therefore, make as much of these as possible.

The next step is the manipulation of the formula or "gaining control of the formula." It has been charged that textbooks

in the past have devoted too much space to puzzle problems and the solution of equations. Nevertheless it would be bad practice to ignore the pedagogical value of the conundrum. In all its varieties—from the riddle to the tragic mystery—it may be a powerful stimulus to intellectual activity. Thus, some authors have a page now and then devoted to recreational material based upon the attraction of the conundrum. This attraction may be used to beguile the pupil into study of processes which are immediately put to a more serious use, that of leading him from old truths to new by infallible mechanical processes. Using certain rules or axioms, any letter which enters into the formula, can be solved for or as Professor Nunn says can be made the "subject of the formula." The first work of this nature need not be based upon any axioms of equality. The teacher can merely take a simple formula like $A = bh$ and show that b would equal A/h from the fact that if a number is the product of two other numbers then when the product and one of those numbers is known, the other number can be found by dividing the product by the known number. Here it may be wise to give a numerical illustration as a basis for the reasoning. The more difficult manipulation of formulas should be deferred until after fractions have been studied.

It should be explained to the pupil that the object of changing the subject or one of them is to save ourselves labor. That a condition might well arise where one should have to find another part of the formula instead of the one solved for first. That if this part was to be sought enough times, it would save time to have it changed to the subject. This is an illustration of the "economy of thought." Some author has said, "Civilization advances by extending the number of important operations which we can perform without thinking about them."

It is my idea that the word equation should be avoided until this point in the course. We have now had practice in formulation and substitution and are ready for work leading up to generalization. This can be done by having tables of numbers arranged in pairs and asking the class to discover the relationship between the various pairs, as

2	5	7
4	25	49

etc. This will illustrate that we do not need to know what the numbers represent

but that the relationship holds just the same. One row can be called "X" and the other "Y." Then it is not hard for a majority of the class to see that $Y = X^2$ expresses the relationship and we have a generalization, which is true for whatever X and Y might be appropriately applied to; as the side and area of a square respectively, etc. Now we may call the generalization an equation and explain the difference between equation and formula on the basis of whether we are just stating a relationship between numbers regardless of what they stand for; or whether we are stating a rule in shorthand for working a certain class of problems. The pupil should generally work out a particular case of the puzzle and should then analyze his solution and put it into symbols. On the other hand if he is able to write down his analysis without previous consideration of a particular case, he should of course be encouraged to do so.

I do not wish to enter into a lengthy discussion of the best method of leading up to rules for changing the subject, whether you call them rules or axioms. There are some who advocate working out rules of procedure from simple numerical illustrations arguing thus: If a boy is told that when seven is added to a certain number the sum is twelve, he will at once state that the number was five. It will not be pretended that he reaches this result by reflecting that if equals be taken from equals the remainders are equal nor that if he could not reach it unaided an appeal to that axiom would help him to conviction. Children can solve such concrete riddles years before they can appreciate the abstract axiom. It is argued that the pupil will see that the solution of this problem "take seven from twelve and you have the other number" is perfectly general, so that whenever he meets "X plus some number equals some other number" he will need to subtract the first given number from the second given one to find the unknown. Similarly for the addition of a number. It seems to me that this just amounts to obtaining rules for transposition, a term which we avoid until later in the course.

Then there is the well known method of the scale pans or balances to teach the operations that may be performed without changing the balance or equation. This method, however, is now questioned by some authorities.

It is not for me to judge but it seems to me that it requires

a more mature mind to arrive at generalizations leading to mere rules while the latter method can be grasped by anyone. This will give the pupil something concrete to reason with for the time being and later he can formulate rules for transposition as we call it at that later time.

When we take up more difficult types of formulas to change the subject, it should be pointed out that a very great use can be made of it. Given the formula

$$C = \frac{nE}{nr + R}$$

which can be obtained from an electrician's hand book. This can be changed to

$$r = \frac{E}{C} - \frac{R}{n} \qquad \text{or} \qquad R = n\left(\frac{E}{C} - r\right).$$

Now the important thing here is that I may be so gravely ignorant of electricity that the original formula is meaningless to me, yet I shall be quite certain that if the original formula was valid, the formulas I have derived from it are equally valid. In this way it is possible for me to discover electrical facts of which my friend the electrician (who though an excellent practical man is, perhaps, but an indifferent algebraist) was actually unaware. Of course it will require his practical knowledge to give meaning to my discoveries.

Another use of the formula may be shown, viz.: that of uniting two fractions with monomial denominators. It can be shown by using simple fractions as

$$\frac{1}{3} + \frac{1}{4} = \frac{4 + 3}{12}$$

that S the sum of the two fractions

$$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$$

is perfectly general and can be used as a formula. To get the sum of a/b and c/d, it should be brought out that the fact that we do not know what words the symbols stand for does not hinder us in the least, for we know that the right hand symbolism

describes the sum of two fractions; a and c being the numerators, b and d the denominators. The steps involved in the arithmetical calculation must therefore be those described by the steps:

$$S = \frac{a}{b} + \frac{c}{d} \qquad \text{or} \qquad S = \frac{ad + bc}{bd}$$

and could be described in words if we wished.

After the study of fractions and uses of parentheses some further time should be spent upon formulas involving such variables as time, rate, and distance; formulas of the levers; more difficult problems of area, perimeter, etc., as a cumulative review to fix the principles involved.

In connection with the trigonometric functions tangent, sine, and cosine which I think we are all agreed should be taught in connection with secondary mathematics, after the ratios have been taught as ratios, I think there is a growing tendency to teach the solution for particular sides of triangles by formula. For instance adjacent side equals hyp. $\times$ cos. of the angle, opp. side equals hyp. $\times$ sin. of the angle etc.

Some advocate calling the trigonometric functions, multipliers or coefficients. It seems to me that it is foolish to have to stop to think out each time just how to solve for the adjacent side when the formula says once for all how it is done.

Later in the course when we reach quadratics there is another very important formula, the value of which the pupils see quickly. It is the one for finding the roots of a quadratic equation. It seems to me that, in conformity to our idea of "economy of thought," it is perfectly proper to allow pupils to solve quadratics by the use of this formula. Of course they should master the technique of the method of completing the square and should develop the formula first.

In closing I think the essentials of a course in algebra can be summed up as follows: The use of the formula as a means of making and expressing arithmetical generalizations, and of describing the quantitative realities which characterize physical, social, and other phenomena. The making of formulas including practice in the simplest forms of algebraic symbolism. The interpretation of formulas and the determination of particular results by substitution. Later on the manipulation of formulas

in order to bring out the further relations which a given generalization may imply. The application of these processes to the solution of problems of real interest and of practical importance.

WATCH THE NATIONAL COUNCIL GROW!

Members of the Council will be interested in the following tabular representation of the mailing list by states for the January and October issues of the MATHEMATICS TEACHER. In every state except fifteen the number of subscriptions has increased. However, this increase is not as great as was that from November 1927 to January 1928 as shown in the February issue. It is hoped that teachers in those states where the membership is low or has actually decreased will do all they can to present the work of the Council at their various meetings. Pamphlets describing the publications of the Council will be sent upon request.

	Jan.	Oct.		Jan.	Oct.
Alabama	76	76	North Carolina	93	101
Arizona	10	15	North Dakota	12	20
Arkansas	26	26	Nevada	2	2
California	157	165	New Hampshire	21	18
Colorado	57	70	New Jersey	124	166
Connecticut	78	90	New Mexico	9	8
Delaware	3	5	New York	568	605
Florida	32	35	Ohio	223	258
Georgia	38	38	Oklahoma	56	103
Idaho	14	13	Oregon	24	34
Illinois	340	346	Pennsylvania	374	396
Indiana	143	169	Rhode Island	24	27
Iowa	99	144	South Carolina	23	26
Kansas	178	166	South Dakota	20	19
Kentucky	34	33	Tennessee	29	38
Louisiana	42	37	Texas	161	167
Maine	30	30	Utah	5	5
Maryland	89	70	Vermont	17	18
Massachusetts	263	289	Virginia	49	57
Michigan	150	190	Washington	33	35
Minnesota	112	121	West Virginia	36	43
Mississippi	39	43	Wisconsin	125	129
Missouri	88	88	Wyoming	14	15
Montana	11	12	Philippine Islands	14	18
Nebraska	74	100	Foreign Countries	103	126

THE TEACHING OF PROPORTION IN PLANE GEOMETRY

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The teaching of proportion offers such difficulties in plane geometry that many teachers of recognized excellence do unsatisfactory work in dealing with this topic. Textbook presentations show wide variation, and it is difficult to say which kind of treatment is better. From the books on the teaching of mathematics one expects help with the pedagogy and presentation of the various aspects of algebra and geometry. But the literature concerned with the teaching of ratio and proportion is pitifully meagre.

The purpose of this article is to report in unified and constructive form the results of a survey of two guides in the teaching of proportion: the professional literature, and the textbooks in plane geometry. Certainly the background provided through the junior high school training is important in its contribution to the study of proportion in demonstrative geometry, and we are giving a brief statement of the help expected from that source. Finally, there is a discussion of the changes which seem advisable in the light of this study.

I. THE LITERATURE ON RATIO AND PROPORTION

In the Elementary School.—There seems to be agreement among authorities that the foundations for the study of ratio and proportion should be laid in the elementary school. Thorndike[1] suggests that the understanding of ratio should begin in the fifth grade with such problems as: "How many times as large (long, heavy, expensive, etc.) as . . . is . . . ?" According to Newcomb,[2] "the teaching of ratio should be begun as early

[1] Edward L. Thorndike, *The Psychology of Arithmetic.* New York: The Macmillan Company, 1922. See pp. 225f.

[2] Ralph S. Newcomb, *Modern Methods of Teaching Arithmetic.* Boston: Houghton Mifflin Company, 1926. P. 228.

as the fifth grade and should constitute one of the important topics of each grade until thoroughly understood."

Of proportion Young[3] says: "Only the simplest phases of proportion seemed to be needed in arithmetic, and they may well be treated informally and untechnically. . . . Proportion need not be made a separate topic. It is simply an application of fractions."

In the Junior High School.—The importance in the junior high school of the concepts *ratio* and *proportion* is indicated by the results of a series of studies reported by Schorling.[4] The position that these terms should be understood through numerous concrete illustrations was unanimously confirmed by curriculum studies, textbooks, practice, the National Committee Report, and a jury of leaders in education and in mathematics.

Smith and Reeve[5] state as objectives in the junior high school the ability to express a ratio graphically, the understanding of a proportion as an equality of ratios, and the understanding that a ratio written in fractional form is subject to all the laws of fractions. They say, further:

> If we are asked to find the ratio of two numbers, we always divide. We may therefore teach that a ratio is a fraction, and include it here (under fractions). . . . According to circumstances, therefore, we may have to think of ratio as a fraction, as a decimal, as a whole number, as a percent, as a quotient, and in many practical cases as a number which we need to use as a multiplier.
>
> A great deal might be gained if, where the word "proportion" is introduced, it were looked upon merely as an equation between two fractions (p. 110).

Regarding ratio in algebra, Thorndike[6] writes:

> Bonds should be made between the facts about ratio and the verbal forms in response to which we use our knowledge of ratio. The most im-

[3] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School.* New York: Longmans, Green and Co., 1924. P. 239.

[4] Raleigh Schorling, *A Tentative List of Objectives in the Teaching of Junior High School Mathematics with Investigations for Determining their Validity.* Ann Arbor: George Wahr, 1925. See p. 101.

[5] David Eugene Smith and William David Reeve, *The Teaching of Junior High School Mathematics.* Boston: Ginn and Company, 1927. See pp. 71, 79f.

[6] Edward L. Thorndike and others, *The Psychology of Algebra.* New York: The Macmillan Company, 1923. Pp. 255f.

portant is: p is . . . as many (much, heavy, long, large, etc.) as q. When p is known to be larger than q, we usually think the word "times" before "as many." The next in importance is "p and q are in the ratio of . . . to," the "fillers" to be made simple for memory's sake and for convenience in calculation.

The third is, "The coefficient of so and so (or the specific gravity, or the nutritive ratio, or the visibility, or what not) is the ratio of the such and such to the such and such." One must know that the first such and such is the numerator, and the second such and such is the denominator. Science should learn to change this usage to "is the such and such divided by the such and such," which would save probably one hundred thousand hours a year to students of algebra and first year science alone. In the meantime, we can save part of it by engraving on every mind "The ratio of *this* to *that* means $\frac{this}{that}$.

Unless these bonds are made, it is hard to learn what ratio means, and still harder to use your knowledge of ratio when it is needed.

In Geometry.—Schultze [7] points out the fact that "the modern school book defines a ratio as a fraction. The finding of a ratio and the determination of a quotient are identical problems. This, however, in many examples involves the notion of irrational numbers, and we can understand that, at a time when only rational numbers were recognized and irrational numbers were considered impossible, the definition of a ratio as a quotient was considered incomplete. Thus Euclid and the other ancient geometers did not use the arithmetical definition of ratio."

Smith [8] calls attention to the introduction of ratio in Book II, in dealing with central angles and their arcs:

> It is then customary (Book II) to define ratio as the quotient of the numerical measures of two quantities in terms of a common unit. This brings all ratios to the basis of numerical fractions, and while it is not scientifically so satisfactory as the ancient concept which considered the terms as lines, surfaces, angles or solids, it is more practical, and it suffices for the needs of the elementary pupils.

In giving pupils an understanding of ratio it is helpful to make actual use of measurement and to compare lines, for example, in terms of their length numbers as measured. Thus Coolidge [9] states that:

[7] Arthur Schultze, *The Teaching of Mathematics in Secondary Schools.* New York: The Macmillan Company, 1912. P. 197.

[8] David Eugene Smith, *The Teaching of Geometry.* Boston: Ginn and Company, 1911. P. 206.

[9] Julian A. Coolidge, "What is a Ratio?" *School Science and Mathematics,* Vol. X, No. 5, May, 1910, pp. 406–9.

With this premise (that the pupil should have a clear notion of the process of measurement), the ratio of two lines is the fraction formed by taking the length number of one line and dividing it by the length number of another line, and so for the ratio of any two comparable objects. It would be worth while to say at this point that there are two or three algebraic manipulations that would be useful, and show them in half a page, and then use them; and it would be instructive to consider the effect of changing the unit of measurement, and note how it will affect ratio. In particular, take the second line as unit; the ratio is then the measure of the first in terms of the second.

However, the chief occasion for concern about ratio and proportion in geometry is at the beginning of Book III, dealing with similar polygons. As Coolidge (*loc. cit.*) describes it ". . . when studying geometry, after passing through the first two books and dealing with matter that is apparently germane to the subject, we come, in the third book, upon a long interlude regarding proportion, full of long words. This takes a lot of time. Is it algebra? If it is algebra, why this hard way of doing easy problems in fractions?"

With regard to the long words and the use of obsolete machinery, Smith [10] says:

The ancients made much of such terms as duplicate, triplicate, alternate, and inverse ratio. These entered into such propositions as, "If four magnitudes are proportional, they will be in proportion alternately." In later works they appear in the form of "proportion by composition," "by division," and "by composition and division." None of these is to-day of much importance, since modern symbolism has greatly reduced the ancient expressions, and in particular the proposition concerning "composition and division" is no longer a basal theorem in geometry. Indeed if our course of study were properly arranged, we might well relegate the whole theory of proportion to algebra, allowing this to precede the work in geometry.

The fact that it should be unnecessary to spend a great deal of time with useless terms and still more useless operations is set forth by Smith and Reeve [11] in this fashion:

It has therefore come to be customary to consider a proportion as a simple type of algebraic equation, abandoning such terms as "alternation" and "inversion," and such statements as that about the product (or rectangle) of the means."

Some teachers have favored . . . breaking away from geometry at this point and spending some time on algebra. . . . It is quite as if, in the

[10] *Op. cit.*, pp. 229f.
[11] *Op. cit.*, p. 263.

work in mensuration, when they wished to use multiplication, with which process the pupil has long been familiar, they should break away from measurements for the purpose of spending some time on arithmetic. The student has long been familiar with such a simple equation as

$$\frac{x}{b} = \frac{c}{d}$$

when he begins his theory of proportional magnitudes in geometry, and he should spend a lesson period in reviewing it with a view to his immediate needs and then proceed with his geometry.

Attention should be directed at this time to the Report of the National Committee on Mathematical Requirements in which it is recommended (p. 76) that the terms *antecedent, consequent, third proportional,* and *fourth proportional* be abandoned.

II. The Junior High School Background

The junior high schools, with their modern courses in mathematics designed specifically to meet newly defined objectives, are doing much to give their pupils a genuine understanding of the principles which serve as preparation for the mathematics of the senior high schools. This is particularly true for ratio and proportion as treated in some of the better known junior high school texts. And there is no doubt that much of the difficulty occasioned in the later study of proportion can justly be ascribed to the inadequate background supplied through courses which are not so well constructed.

In junior high schools giving these modern courses, ratio and proportion comprise a large unit of instruction which runs through all three years of the mathematics work. Long before the formal terms and principles are used, the pupil is given numerous exercises and vivid illustrations of the ideas involved. Even before the term *ratio* is used there are many illustrations of the notion, and the growing concept is related to the measurement of line segments, to common fractions, to decimal fractions, and to percents. A great deal of practice is given, after the term is introduced, in using ratio as a means of comparing quantities, especially in working with line, bar, and circle graphs.

Then, in working with scale drawings and similar triangles, simple proportions occur as *equations* expressing the equality of two ratios. The numerical trigonometry now included in

junior high school courses offers an excellent basis for practice in dealing with ratios and proportions.

Proportion is still further illustrated and applied in the junior high school in dealing with variation and with lever problems. This includes the graphing of direct and indirect variation.

By this time the pupil has become familiar with the term *proportion* and has learned that a proportion is subject to the laws applying to equations in general. In many cases he learns to make one or two transformations as a matter of applying the axioms of algebra. Usually he has learned to make habitual use of the fact that the product of the means equals the product of the extremes.

III. An Analysis of the Textbooks

The practice of textbook writers in teaching the pupil the facts of proportion and how to use them shows considerable variation. In order to analyze the nature and extent of these variations, we have examined the presentation of the theory of proportion in ten of the better known textbooks which have been published or revised since the National Committee on Mathematical Requirements issued its authoritative report. The authorship of the books is indicated in the note accompanying the first table.

In view of the recommendations of the report and of various individual authorities on the teaching of mathematics, an important consideration in this analysis is the occurrence and definition of terms peculiar to proportion. The following table lists these terms and for each book cites the page on which the term is defined.

The traditional practice of giving the various properties of proportion as theorems to be proved by algebraic processes is beginning to weaken under the criticism of progressive authorities. Some of our forward-looking authors, in order to make the uses of proportion in geometry less difficult of acquisition and more effective in application, have seen this topic as a natural development from algebra, and have expressed it informally, working from the algebraic to the verbal forms. Hence, we are interested in the extent to which the traditional practice is still followed by textbook writers. This is shown in Table

TABLE I

Occurrence of Certain Definitions in Ten Geometries

Definitions	Page of Occurrence										No. Bks.
	Av.	C-O	D-A	F-A	PTF	Sey	Smi	S-R	W-H	Y-S	
Ratio { Book II			138	121		153	129	123	114		6
Ratio { Later	181	203			199	201			151	149	6
Proportion { Book II			138								1
Proportion { Later	182	209	178	142	240	201	157	180	152	173	10
Means, extremes	183	209	178	142	240	201	157	182	152	173	10
Mean proportional	183	209	178	160	240	202	157	182	153	173	10
Alternation	*	211	181	144	242	205	159	187	154		8
Inversion	*	211	181	143	242	206	159	187	154		8
Fourth proportional	183	216	178	153		202		182	152	173	8
Antecedent, consequent	183	209		142		201	157	182	151		7
Addition, subtraction		211	182		244	206		188	154		6
Third proportional	183			153		202		182			4
Composition, division	*			144		*	159				2

Note:—The books indicated are : Avery, Clark-Otis, Durell-Arnold, Ford-Ammerman, Palmer-Taylor-Farnum, Seymour, Smith, Strader-Rhoads, Wells-Hart, and Young-Schwartz. The asterisks in the table designate terms mentioned in informational notes.

TABLE II

METHODS OF PRESENTING THE THEORY OF PROPORTION IN TEN GEOMETRIES

	Av.	C–O	D–A	-A	PTF	Sey	Smi	S–R	W–H	Y–S
Presentation	T	ExP	T	T	T	T	A.V	T	T	P
Number of pages	3	1–1	6	3	4	4	1–1	3	3	$\frac{1}{4}$
Number of exercises	11	60	50	6	25	16	30	25	25	5
Location of exercises	F	F	D	F	F	D	F	D	D	F

Abbreviations: T, theorems; ExP, exercises and properties; A.V, algebraic and verbal; D, distributed; F, following. For names of books, see note to Table I.

TABLE III

ITEMS OF THE THEORY OF PROPORTION PRESENTED IN TEN GEOMETRIES

Items	Page of Presentation										No. Bks.
	Av.	C–O	D–A	F–A	PTF	Sey	Smi	S–R	W–H	Y–S	
1. $a/b = c/d;\ ad = bc$	183	211	179	143	241	203	159	183	153	173	10
2. $ad = bc;\ a/b = c/d$	184	211	180	143	242	203	159	184	153		9
3. $a/b = c/d;\ a/c = b/d$	ex.	211	181	144	242	205	159	187	154		8
4. $a/b = c/d;\ b/a = d/c$	ex.	211	181	143	242	206	159	187	154		8
5. $a/b = c/d;\ a + b/b = c + d/d$	ex.	211	182	144	242	206	159	188	154		8
6. $a/b = c/d = e/f;\ a + c + e/b + d + f = a/b$	182	211	183	145	242	204	159		181		8
7. $a/x = a/y;\ x = y$	184	211	180	143	241		159	184	154		8
8. $a/b = b/c;\ b = \sqrt{ac}$	184		179		241	204	ex.	183	153	173	7
9. $a/b = c/d;\ a - b/b = c - d/d$	ex.	211		144	242	206	159	188	154		7
10. $a/b = c/x;\ a/b = c/y;\ x = y$		211	180		241	203		184	154		6
11. $a/b = c/d;\ a + b/a - b = c + d/c - d$				144	242				154		3
12. $a/b = c/d;\ a^n/b^n = c^n/d^n$			184		242	207	ex.				3
13. $a/b = c/d;\ a^{1/n}/b^{1/n} = c^{1/n}/d^{1/n}$			184		242	207	ex.				3
14. $a/b = c/d;\ e/f = g/h;\ j/k = l/m;\ aej/bfk = cgl/dhm$			183			204					2

II. The table also shows the number of exercises giving practice in the uses of proportion before the study of geometry is continued, and indicates whether these exercises are distributed through the presentation or follow it.

No small part of the variation in the way proportion is taught arises from the phases of the topic given in the different books. In an effort concisely to show which authors include each of a number of items in the theory of proportion, we have expressed the principles algebraically and ranked them according to frequency of use. The results of this analysis appear in Table III.

Of the ten books examined Durell and Arnold, and Young and Schwartz take the extreme positions. Durell and Arnold devote six pages to the theory of proportion, presenting twelve of the fourteen properties given in the accompanying table, with quasi-geometric proof for each. Our check indicates that only numbers 1, 3–6, and 10 (see table) are used in subsequent theorems. Young and Schwartz on about one-fourth of a page list only two of the properties. No proof is given for either, although the algebraic illustrations are supplied. When a transformation becomes necessary in the proof of a theorem, the required algebraic steps are included in the demonstration. For example, whenever a transformation of "addition" becomes necessary in a proof, 1 is added to each member of the equation.

The Smith and Clark-Otis texts both have two pages on the properties of proportions. Smith gives first a brief algebraic statement of each property, merely suggesting the method of proof. On the next page he gives the properties as verbal statements, each of which is to be proved as an exercise. The Clark-Otis treatment is similar to that of Smith. However, an important psychological difference is that Clark and Otis first present the properties as things to be done by the pupil, algebraic statements to be examined informally, before they are stated verbally. The pupil is then asked to illustrate each of the verbal statements by an equation.

Four of the books, Clark-Otis, Young and Schwartz, Palmer-Taylor-Farnum, and Strader and Rhoads, introduce proportion after a study of areas. The other texts, with one exception, first use the term *ratio* in the second book, in connection with circles; and they elaborate the theory of proportion at the beginning of the third book.

IV. Proposed Changes

A number of the changes in the teaching of proportion, recommended by our foremost authorities in the teaching of mathematics and widely recognized as being desirable, are yet to be accomplished. However, it is well to subject changes in practice to careful study and rigorous experimentation under such circumstances that values can be proved and gains demonstrated: "humanity is better served by nature's gradual changes than by earthquakes."

During the seventeen years since the publication of Smith's *Teaching of Geometry* there has been an increasing tendency to modify the earlier complicated presentation of the theory of proportion in the direction of the simple treatment now exemplified in the recent geometries by Smith, and Clark and Otis. No doubt the "vested interests," in particular teachers of mathematics, have made it necessary to introduce these changes quite gradually, and even now some of the textbooks cling rather tenaciously to the vanishing forms. Thus, progressive authors frequently find it advisable in following accepted recommendations to call attention to disappearing usage by means of informational notes.

At any rate, the tendency in geometry seems to be very definitely toward restriction of detailed study of the theory of proportion. The development of algebraic symbolism since the sixteenth century has made it possible and legitimate to treat algebraically the theorems in proportion which the ancient geometers had to demonstrate by pure geometry. Hence, the theory of proportion has become a matter of algebra, not geometry. The algebraic treatment of the topic as given in many textbooks on geometry is extraneous to demonstrative geometry and is justifiable only insofar as it is necessary for adequate presentation of the geometric materials. In geometry the purpose of the treatment, then, should be not primarily to teach the theory of proportion, but to enable the pupil to master the geometric materials concerned. This being the case, it is a psychologically sound premise that the various needed aspects of proportion should be considered as the need arises. Thus there would be no occasion for the pupil's mastering the topic with all its ramifications, long words, and unnecessary algebraic manipulations before he is allowed to begin his study of proportionality in

similar polygons. In fact, in consideration of meaningful situations and illustrations, it is highly desirable that the useful facts of proportion be reviewed or presented in conjunction with the problems to which they apply.

We are inclined to believe that further improvement in the teaching of proportion in geometry can be effected by a considerable reduction of the isolated presentation of the topic, with a wider use in demonstrations of the axioms which form the basis of solving algebraic equations. We feel that the pupil can gain the essential concepts with richer understanding and more lasting insight if they are presented through many concrete illustrations in various situations than if they are expressed through verbal statements with single algebraic illustrations.

We are about to subject to careful study a treatment of proportion based upon the foregoing hypothesis. It will introduce the needed ideas and processes in connection with the problems in which they are to be used, and will present series of exercises which provide rich experiential background for consideration of the developments involved.

NATIONAL COUNCIL MEMBERS IN CITIES

It is clear that many teachers in the larger cities of the country are not members of the Council. For the benefit of members who desire to help us reach our goal of 10,000 members by 1930 we present the names of some of the larger cities below together with the number of council members in each.

Atlanta, Ga.	12	Minneapolis, Minn.	23
Baltimore, Md.	31	Nashville, Tenn.	6
Birmingham, Ala.	27	Newark, N. J.	11
Boston, Mass.	27	New Orleans, La.	5
Chicago, Ill.	87	New York, N. Y.	216
Cleveland, Ohio	47	Omaha, Neb.	8
Denver, Col.	25	Philadelphia, Pa.	115
Detroit, Mich.	40	Portland, Me.	5
Indianapolis, Ind.	27	Providence, R. I.	21
Kansas City, Mo.	13	Richmond, Va.	2
Los Angeles, Cal.	17	San Antonio, Texas	34
Louisville, Ky.	9	Seattle, Wash.	9
Milwaukee, Wis.	18	Washington, D. C.	37

STEWART'S THEOREM, WITH APPLICATIONS

BY PROFESSOR RICHARD MORRIS

Rutgers University, New Brunswick, N. J.

The writer will introduce this paper by quoting a sentence from Cajori's *History of Mathematics.* "While in France the school of G. Monge was creating Modern Geometry, efforts were made in England by Robert Simson (1687–1768) and Matthew Stewart (1717–1785) to revive Greek Geometry. The latter was a pupil of Simson and was one of the two prominent mathematicians in Great Britain during the eighteenth century." Dr. David Eugene Smith speaks of Euclid as the most influential text-book on mathematics ever written. We are grateful to these two men, Simson and Stewart, for their labors in the Renaissance of Geometry.

Let me quote a sentence also from *Euclidean Geometry*, by Henry George Forder, a book published in 1927. "The fruitless efforts to prove the Parallel Axiom from the other assumptions of Euclid resulted at length in the creation of the non-Euclidean Geometry of Bolya and Lobatschefsky—a landmark in the history of human thought." A reverence for the past should not stagnate our ambitions for progress, neither should an obsession for research dull our gratitude for our legacy. The writer thus feels that it may be worth while to look at a theorem that goes by the name of Stewart.

Had Stewart not ignored the methods of higher analysis, undoubtedly he would have achieved results as important as those obtained by others in the field of analysis. In an article by J. S. Mackay, M.A., LL.D., in the *Proceedings of the Edinburgh Mathematical Society*, Vol. X, 1891–1892, p. 90, we learn that Stewart was the first to publish anything relative to this theorem, 1746, but that Robert Simson had solved the problem earlier, 1741, who however had not published his solution till 1749.

Chronology of Proofs and Publishing.

Robert Simson, solved 1741,	published	1749.
Matthew Stewart,	"	1746.
Thomas Simpson,	"	1752.
—— Euler,	"	1780.
—— Carnot,	"	1803.

Stewart's Theorem. *A line from the vertex of a triangle to any point of the base divides the base into two segments such that the base times the square of the line equals one side squared times the remote segment of the base plus the square of second side times its remote segment minus the product of the base and its two segments.*

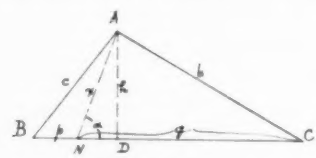


Fig. 1.

Definition: We shall define a *Stewart line* as the segment from a vertex to any point of the opposite side. There are thus many notable lines which are Stewart lines.

Symbolic Statement of Stewart's Theorem (see Fig. 1):

$$a(AN)^2 = pb^2 + qc^2 - apq,$$

or

$$\mathbf{x}^2 = \frac{\mathbf{pb}^2 + \mathbf{qc}^2}{\mathbf{p} + \mathbf{q}} - \mathbf{pq}. \tag{1}$$

We shall give three well-known proofs.

First Proof.

$$x^2 = h^2 + (ND)^2, \tag{1}$$

$$b^2 = h^2 + (q - ND)^2, \tag{2}$$

$$c^2 = h^2 + (p + ND)^2. \tag{3}$$

Subtract (2) from (1) and (3) from (1). Upon eliminating $2pqND$, we obtain formula (1) as given above.

Second Proof (by projection). $c^2 = x^2 + p^2 + 2p(ND)$ and $b^2 = x^2 + q^2 - 2q(ND)$. Again we get the same formula by eliminating $2pq(ND)$.

Third Proof (using Trigonometry). $c^2 = x^2 + p^2 + 2px \cos \alpha$ and $b^2 = x^2 + q^2 - 2qx \cos \alpha$. Hence, by eliminating $2pqx \cos \alpha$, there is obtained the desired formula.

Theorem of Apollonius (260–200) B. C. *If N is the midpoint of BC, $p = q = \frac{1}{2}a$ and x is the length of the median, M_1,* then

$$\mathbf{M}_1^2 = \frac{\mathbf{b}^2 + \mathbf{c}^2}{2} - \frac{\mathbf{a}^2}{4}. \tag{2}$$

Trivial cases. When N coincides with B, or C or D, x becomes equal to c or b or $c \sin B$ or $b \sin C$.

Other Positions of N. If N is anywhere on BC produced, say to the left of B, then p is negative.

Formula (1) then becomes

$$\mathbf{x}^2 = \frac{\mathbf{qc}^2 - \mathbf{pb}^2}{\mathbf{q} - \mathbf{p}} + \mathbf{pq}. \tag{3}$$

If however, N is to the right of C, then q is negative and we get from (1)

$$\mathbf{x}^2 = \frac{\mathbf{pb}^2 - \mathbf{qc}^2}{\mathbf{p} - \mathbf{q}} + \mathbf{pq}. \tag{4}$$

In this paper we shall assume $a > b > c$ as the relative lengths of the sides of the triangle.

Problem. *The shortest median is drawn to the longest side.*

$$M_1^2 < M_2^2 \quad \text{since} \frac{b^2 + c^2}{2} - \frac{a^2}{4} < \frac{a^2 + c^2}{2} - \frac{b^2}{4}$$

and

$$M_2^2 < M_3^2 \quad \text{since} \frac{a^2 + c^2}{2} - \frac{b^2}{4} < \frac{a^2 + b^2}{2} - \frac{c^2}{4}.$$

Inclination of a Stewart line to its side. Let h denote the altitude AD and α the angle ANC. Then

$$\cot \alpha = \frac{ND}{h}$$

and

$$\begin{aligned}(p + q) \cot \alpha &= \frac{p(ND)}{h} + \frac{q(ND)}{h} \\ &= p\frac{(q - DC)}{h} + q\frac{(BD - p)}{h} \\ &= \frac{pq}{h} - p\frac{DC}{h} + q\frac{BD}{h} - \frac{pq}{h}.\end{aligned}$$

Hence

$$(\mathbf{p} + \mathbf{q}) \cot \alpha = \mathbf{q} \cot \mathbf{B} - \mathbf{p} \cot \mathbf{C}. \tag{5}$$

If N coincides with D, $\cot \alpha = 0$, and when N coincides with the mid-point, we get from (*5*),

$$\cot \alpha = \frac{\cot \mathbf{B} - \cot \mathbf{C}}{2}. \tag{6}$$

Let β and γ denote the inclinations of the other medians to their respective sides, then

$\cot\beta = \frac{1}{2}(\cot C - \cot A)$ and $\cot\gamma = \frac{1}{2}(\cot A - \cot B)$,

whence

$$\cot\alpha + \cot\beta + \cot\gamma = 0.$$

If p_1 and q_1, p_2 and q_2, and p_3 and q_3 denote the segments of sides BC, CA and AB respectively, then

$$\begin{aligned}(p_1 + q_1)\cot\alpha + (p_2 + q_2)\cot\beta + (p_3 + q_3)\cot\gamma \\ = q_1\cot\beta - p_3\cot B + q_2\cot C - p_1\cot C + q_3\cot A \\ - p_2\cot A.\end{aligned}$$

If however,

$$\frac{pi}{qi} = \frac{m}{n},$$

we get

$$\cot\alpha + \cot\beta + \cot\gamma = \frac{n-m}{n+m}[\cot A + \cot B + \cot C].$$

Geometric Proof for Formula (6). Draw a parallelogram with triangle ABC as one-half, such as $ABA'C$ (see Fig. 5) and drop perpendiculars from A and A' to BC, denoting their feet by D and D' respectively. Then $AD'A'D$ is also a parallelogram and $BD = D'C$. Hence

$$DN = \tfrac{1}{2}(DC - BD)$$

and

$$\cot(180 - \alpha) = \frac{DN}{AD} = \frac{1}{2}\left(\frac{DC}{AD} - \frac{BD}{AD}\right), \qquad [\alpha = ANC$$

$$= \tfrac{1}{2}(\cot C - \cot B),$$

or

$$\cot\alpha = \tfrac{1}{2}(\cot B - \cot C),$$

since angle $B >$ angle C and $\alpha > 90°$.

Other Forms of Theorem (1).

I. Let $\frac{p}{q} = \frac{m}{n}$, then $p = \frac{am}{m+n}$ and $q = \frac{an}{m+n}$.

Substituting these values of p and q in (1) we get

$$\mathbf{x}^2 = \frac{\mathbf{mb}^2 + \mathbf{nc}^2}{\mathbf{m} + \mathbf{n}} - \frac{\mathbf{mna}^2}{(\mathbf{m} + \mathbf{n})^2}. \tag{7}$$

This formula will serve when BC is divided into $m + n$ equal parts, m and n being positive integers.

Example. Let $a = 9$, $b = 8$, $c = 6$ and $p : q = 4 : 5$. Find $x = 5\frac{1}{3}$.

II. Again, let $q = np$, then $p = \dfrac{a}{n+1}$, $q = \dfrac{an}{n+1}$, and substituting in (1) we get

$$\text{(8)} \qquad \mathbf{x^2 = \frac{b^2 + nc^2}{n+1} - \frac{na^2}{(n+1)^2}}.$$

This formula applies when BC is divided into $n + 1$ equal parts.

Example. Let $n = 2$ in (8) and get $b^2 + 2c^2 = 3x^2 + \frac{2}{3}a^2$, where x is the Stewart line to the point of trisection nearer B. Let y denote the Stewart line to the point of trisection nearer C and get $x^2 + y^2 = b^2 + c^2 - \frac{4}{9}a^2$. Let C be a right angle, then $x^2 + y^2 = \frac{5}{9}a^2$. Find $x^2 + y^2$ if the triangle is equilateral.

A General Theorem. *Let BC be divided into n equal parts. Find the sum of the squares of the Stewart lines to these points of division, excluding points B and C.* There will be two cases, depending upon whether n is even or odd. (See Fig. 2.)

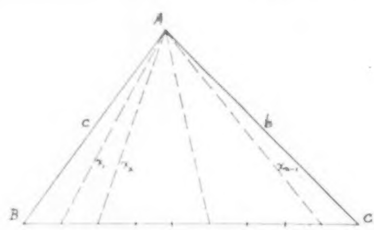


FIG. 2.

Applying formula (1), $(n - 1)$ times in succession, we get the following series of equations.

$$x_1^2 = \frac{\dfrac{a}{n}b^2 + \dfrac{n-1}{n}ac^2}{a} - \frac{1(n-1)a^2}{n^2},$$

$$x_2^2 = \frac{\dfrac{2}{n}ab^2 + \dfrac{n-2}{n}ac^2}{a} - \frac{2(n-2)a^2}{n^2},$$

$$x_3^2 = \frac{\dfrac{3}{n}ab^2 + \dfrac{n-3}{n}ac^2}{a} - \frac{3(n-3)a^2}{n^2},$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$x_{n-1}^2 = \frac{\frac{n-1}{n}ab^2 + \frac{a}{n}c^2}{a} - \frac{(n-1)\,a^2}{n^2}.$$

Whence

$$(A)\quad x_1^2 + x_2^2 + \cdots + x_{n-1}^2$$
$$= \frac{1+2+3+\cdots+(n-1)}{n}(b^2+c^2)$$
$$- \frac{a^2}{n^2}[1(n-1) + 2(n-2) + 3(n-3) + \cdots$$
$$+ (n-1)1].$$

Case I (n odd). *If n is odd, there will be $(n-1)$ points excluding B and C, i.e. an even number.* The coefficients of b and c^2 are equal, each being $\frac{n-1}{2}$, the numerator being an arithmetic series.

The series in the brackets of (A) consists of two equal parts of $\frac{n-1}{2}$ terms each, and may be written,

$$2\left[1(n-1) + 2(n-2) + 3(n-3) + \cdots + \frac{n-1}{2}\left(n - \frac{n-1}{2}\right)\right]\frac{a^2}{n^2},$$

or

$$2\left[n + 2n + 3n + \cdots + \frac{n-1}{2}n - \left\{1 + 4 + 9 + \cdots + \left(\frac{n-1}{2}\right)\right\}\right]\frac{a^2}{n^2},$$

or

$$2\left[\frac{n(n^2-1)}{8} - \frac{n(n^2-1)}{24}\right]\frac{a^2}{n^2},$$

using the formula for the sum of an arithmetical series and the formula for the sum of the squares of the first $\frac{n-1}{2}$ consecutive integers, whence by further simplification we get $\frac{(n^2-1)a^2}{6n}$.

Thus we have

$$(9)\quad \mathbf{x_1^2 + x_2^2 + \cdots + x^2_{n-1} = \frac{n-1}{2}(b^2+c^2) - \frac{(n^2-1)a^2}{6}}.$$

Case 2 (n even). *If n is even, there will be $(n-1)$ points, excluding B and C, i.e. an odd number.* As before, the coefficients of b^2 and c^2 are each equal to $\frac{n-1}{2}$.

The series in the brackets of (A) consists of a middle term and two equal parts. The number of the middle term is $\frac{n}{2}$ and the number of terms in each of the equal series is $\left(\frac{n}{2}-1\right)$. Hence the series in brackets becomes

$$2\left[1(n-1)+2(n-2)+\cdots+\left(\frac{n}{2}-1\right)\left\{n-\left(\frac{n}{2}-1\right)\right\}\right]\frac{a^2}{n^2}+\frac{n}{2}\left(n-\frac{n}{2}\right)\frac{a^2}{n^2},$$

or

$$2\left[n+2n+\cdots+\left(\frac{n}{2}-1\right)n-\left\{1+4+9+\cdots+\left(\frac{n}{2}-1\right)\right\}\right]\frac{a^2}{n^2}+\frac{a^2}{4},$$

or

$$2\left[\frac{n^2(n-2)}{8}-\frac{n(n-1)(n-2)}{24}\right]\frac{a^2}{n^2}+\frac{a^2}{4},$$

and finally

$$\frac{(n-2)(2n+1)}{12}\cdot\frac{a^2}{n}+\frac{a^2}{4}.$$

Thus we have,

$$(10)\quad x_1^2+x_2^2+\cdots x_{n-1}^2 = \frac{n-1}{2}(b^2+c^2)-\left[\frac{(n-2)(2n+1)}{12}\cdot\frac{a^2}{n}+\frac{a^2}{4}\right].$$

Example. Let $n=3$ and find $x_1^2+x_2^2=b^2+c^2-\frac{4}{9}a^2$.

Example. Let $n=4$ and find $x_1^2+x_2^2+x_3^2=\frac{3(b^2+c^2)}{2}-\frac{5}{8}a^2$.

Theorem I. *Of two Isotomic-conjugate Stewart lines*[1] *emanating from the same vertex of a triangle, the greater Stewart line is nearer the greater of the two sides of the triangle intersecting at the vertex.*

[1] See The Mathematics Teacher for March 1928 for a paper on Isotomic Lines and Points.

Let x_1 and x_2 be the internal pair of lines (see Fig. 3) and x_3 and x_4 the external pair. Then $x_1^2 = \dfrac{pb^2 + qc^2}{a} - pq$ and $x_2^2 = \dfrac{qb^2 + pc^2}{a} - pq$. Now x_2^2 will be greater than x_1^2 if

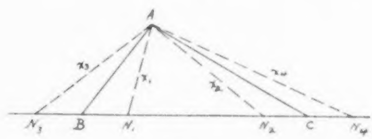


FIG. 3.

$qb^2 + pc^2 > pb^2 + qc^2$. But $(q - p)b^2$ is greater than $(q - p)c^2$ since b is greater than c by assumption. Again

$$x_3^2 = \frac{qc^2 - pb^2}{a} + pq \quad \text{and} \quad x_4^2 = \frac{qb^2 - pc^2}{a} + pq,$$

and x_4^2 will be greater than x_3^2 if $qb^2 - pc^2$ is greater than $qc^2 - pb^2$. But $(q + p)b^2$ is greater than $(q + p)c^2$ since $b > c$ by assumption. If in Theorem I, we set $p = d < \dfrac{a}{2}$ and $q = a - d$ for X_1 and X_2; and let $p = d$ and $q = a + d$ for x_3 and x_4, we prove

Theorem II.

$$x_2^2 \lesseqqgtr x_3^2 \quad \text{if} \quad d \gtreqqless \frac{b^2 - c^2}{2a}.$$

The equality case is easily shown by substitution, the other cases are a little more difficult. It can be shown that x_3^2 is greater than x_2^2 if $\dfrac{b^2 - c^2}{2a}$ is increased by a quantity e where e is such that $\dfrac{b^2 - c^2}{2a} + e$ is less than $\dfrac{a}{2}$.

For x_2,

$$p = a - \frac{b^2 - c^2}{2a} - e \quad \text{and} \quad q = \frac{b^2 - c^2}{2a} + e.$$

For x_3,

$$p = a + \frac{b^2 - c^2}{2a} + e \quad \text{and} \quad q = \frac{b^2 - c^2}{2a} + e.$$

When these values of p and q are substituted in the values of x_2^2 and x_3^2 in Theorem I, we find the value of $x_3^2 - x_2^2 = 2ae$

which is a positive quantity. Similarly $\frac{b^2 - c^2}{2a}$ is diminished by e, it can be shown that x_2^2 is greater than x_3^2. In this case, for x_2,

$$p = a - \frac{b^2 - c^2}{2a} + e \quad \text{and} \quad q = \frac{b^2 - c^2}{2a} - e\ ,$$

and for x_3,

$$p = \frac{b^2 - c^2}{2a} - e \quad \text{and} \quad q = a + \frac{b^2 - c^2}{2a} - e\ .$$

And we get $x_2^2 - x_3^2 = 2ae$.

For a set of four Stewart lines as in Theorems I and II, there is the following series of inequalities.

(a) $$x_4^2 > x_3^2 > x_2^2 \left(\text{if } d > \frac{b^2 - c^2}{2a} \right) > x_1^2,$$

(b) $$x_4^2 > x_3^2 = x_2^2 \left(\text{if } d = \frac{b^2 - c^2}{2a} \right) > x_1^2,$$

(c) $$x_4^2 > x_2^2 > x_3^2 \left(\text{if } d < \frac{b^2 - c^2}{2a} \right) > x_1^2.$$

Example. Let $a = 12$, $b = 10$ and $c = 6$.

Then $d = 2\frac{2}{3}$. For a simple case, let $e = \frac{1}{3}$ for case (a) and $e = \frac{2}{3}$ for case (c). We tabulate the three cases.

(a) $x_4^2 = 161$, $x_3^2 = 65$, $x_2^2 = 57$, $x_1^2 = 25$.

(b) $x_4^2 = 153\frac{1}{3}$, $x_3^2 = 60\frac{8}{9} = x_2^2$, $x_1^2 = 25\frac{1}{3}$.

(c) $x_4^2 = 138\frac{2}{3}$, $x_3^2 = 53\frac{1}{3}$, $x_2^2 = 69\frac{1}{3}$, $x_1^2 = 26\frac{2}{3}$.

When used additively, e lies between 0 and $\frac{a^2 - b^2 + c^2}{2a}$, but when used subtractively, it lies between 0 and $\frac{b^2 - c^2}{2a}$.

Special Stewart Lines.

(a) **Internal angle bisector of angle A.** We have $p : q = c : b$, hence

$$p = \frac{ac}{b + c} \text{ and } \quad q = \frac{ab}{b + c}\cdot$$

Using (1) we get

$$x^2 = \frac{bc(b^2 + 2bc + c^2 - a^2)}{(b + c)^2}$$

or

$$x = \frac{1}{b + c}\sqrt{bc(a + b + c)(b + c - a)} = \frac{2bc}{b + c}\sqrt{\frac{s(s - a)}{bc}},$$

where $2s = a + b + c$.

Example. Apply Stewart's Theorem to the Isotomic conjugate of internal bisector.

(*b*) **External angle bisector of angle A.** In this case

$$p = \frac{ac}{b - c} \text{ and } \quad q = \frac{ab}{b - c}.$$

Using (3), we get

$$x^2 = \frac{bc}{(b - c)^2}[a^2 - (b - c)^2]$$

and

$$x = \frac{2bc}{b - c}\sqrt{\frac{bc}{(s - b)(s - c)}}.$$

Example. Apply Stewart's Theorem to the Isotomic-conjugate of the external bisector.

(*c*) **The Symmedian for angle A.** The Stewart line making the same angle with the internal angle bisector as the median is called a symmedian. In this case $p = \dfrac{ac^2}{b^2 + c^2}$ and $q = \dfrac{ab^2}{b^2 + c^2}$.[1] Hence,

$$(\text{Symmedian})^2 = \frac{2b^2c^2}{b^2 + c^2} - \frac{a^2b^2c^2}{(b^2 + c^2)^2}.$$

Example. Show that Symmedian : median $= 2bc : b^2 + c^2$.

Remark. Evidently the Symmedian is always less than its median.

Example. Apply Stewart's Theorem to the Isotomic-conjugate of the symmedian and show that the Symmedian : Isotomic-conjugate $= 2b^2c^2 : b^4 + c^4$.

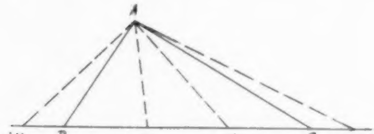


FIG. 4.

In Fig. 4 let N_1, N_2, N_3 and N_4 be the points on BC of tangency of the in-circle, the ex-circle opposite angle A, the ex-circle opposite C and the ex-circle opposite B respectively.

[1] See Court, *College Geometry*.

(*d*) **Stewart lines to points of tangency.** For line AN_1, $p = s - b$ and $q = s - c$. Hence

$$(AN_1)^2 = \frac{(s-b)b^2 + (s-c)c^2}{a} - (s-b)(s-c).$$

For line AN_2, $p = s - c$ and $q = s - b$. Hence

$$(AN_2)^2 = \frac{(s-c)b^2 + (s-b)c^2}{a} - (s-b)(s-c).$$

Theorem I states that $(AN_2)^2$ is greater than $(AN_1)^2$ since $s - b$ is less than $s - c$. But this may also be established by comparing the numerators of the fractions for AN_2 and AN_1.

We have $sb^2 + sc^2 = sb^2 + sc^2$. But

$$bc(b + c) < (b^2 - bc + c^2)(b + c),$$

since

$$bc < b^2 - bc + c^2.$$

Hence

$$sb^2 + sc^2 - cb^2 - bc^2 > sb^2 + sc^2 - b^3 - c^3$$

or

$$(s-c)b^2 + (s-b)c^2 > (s-b)b^2 + (s-c)c^2.$$

Hence

$$(AN_2)^2 > (AN_1)^2.$$

That is, the internal Nagel line is greater than the internal Gergonne line.[2] For line AN_3, $p = s - a$ and $q = s$. Hence,

$$(AN_3)^2 = \frac{sc^2 - (s-a)b^2}{a} + s(s-a).$$

For line AN_4, $p = s$ and $q = s - a$. Hence

$$(AN_4)^2 = \frac{sb^2 - (s-a)c^2}{a} + s(s-a).$$

As before, by comparing numerators, we can show that

$$(AN_4)^2 > (AN_3)^2.$$

We have $2s > a$ and

$$2s(b^2 - c^2) > a(b^2 - c^2)$$

or

$$sb^2 - sc^2 + sb^2 - sc^2 > ab^2 - ac^2$$

or

$$sb^2 - sc^2 + ac^2 > sc^2 - sb^2 + ab^2$$

[1] See MATHEMATICS TEACHER, March 1928.

or

$$sb^2 - (s-a)c^2 > sc^2 - (s-a)b^2.$$

Hence

$$AN_4 > AN_3.$$

Example. If $a = 12$, $b = 10$, $c = 6$, the square of the Stewart lines to points of tangency on BC are as follows:

$$(AN_1)^2 = 25\tfrac{1}{3}, \qquad (AN_2)^2 = 46\tfrac{2}{3}, \qquad (AN_3)^2 = 53\tfrac{1}{3} \quad \text{and} \quad (AN_4)^2 = 138\tfrac{2}{3}.$$

Example. If $a = 12$, $b = 10$, $c = 4$, we get 10, 52, 22 and 120 as the values of the squares of the respective lines. We notice in these two cases an interchange in the relative values of the lines AN_2 and AN_3.

Theorem III.

$$(AN_3)^2 \gtreqless (AN_2)^2 \quad \text{if} \quad c \gtreqless \frac{\sqrt{4b^2 + a^2} - a}{2}.$$

Let $(AN_3)^2$ be set equal to $(AN_2)^2$. After simplifying, we get

$$\frac{b}{a}(c^2 + ac - b^2) = 0,$$

and hence

$$c = \frac{\sqrt{4b^2 + a^2} - a}{2} = D.$$

If $(AN_3)^2$ is to be greater than $(AN_2)^2$, then $\frac{b}{a}(c^2 + ac - b^2) > 0$. But since a and b are given, we get $c > D$. Similarly, we get $c < D$ if $(AN_3)^2 < (AN_2)^2$.

We note that for given values of a and b, c may be constructed, using a right triangle whose legs are $2b$ and a.

Note also that Theorems I and II relate to variations of d when the sides of the triangle are given, while Theorem III relates to changes in c when a and b are given.

In the cases of Stewart lines from vertex A to the points of tangency on line BC, distances BN_1 and CN_2 are equal and BN_3 equals CN_4. But the four distances are not in general equal; this can be true only when $a = b$, since $s - a$ must equal $s - b$ in that case. All of the external segments on the sides of the triangle will be equal only in the case of an equilateral triangle, since $s - a = s - b = s - c$ in this case. In this last case, the number of Stewart lines reduces from 12 to 9, because of coincidence with medians, altitudes and internal angle bisectors.

The Parallelogram.

Let $ABA'C$ be a parallelogram constructed on the triangle ABC. (See Fig. 5.) Using (1) or (2) for triangles ABC and $A'BC$ we get $(2x)^2 + a^2 = 2(b^2 + c^2)$. Hence the well known

Theorem IV. *The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides.*

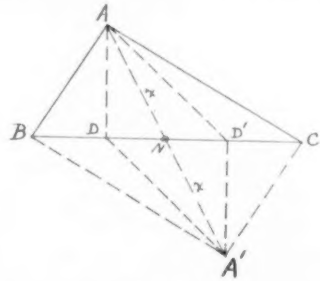


Fig. 5.

Problem. *Apply Theorem IV to an oblique parallelepiped to show that the sum of the squares of the four diagonals equals the sum of the squares of the twelve (12) edges.*

Problem. *Apply Theorem IV to a tetrahedron to show that the sum of the squares of the three lines (medians) joining the mid-points of the opposite edges is equal to one-fourth of the sum of the squares of the six (6) edges.*

The Trapezium.

In Fig. 6 it is easily shown that AA', BB' and CC', mid-points of sides and diagonals, are concurrent in point O.[1] Let

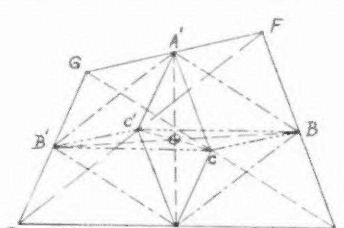


Fig. 6.

$DE = a$, $EF = b$, $FG = a'$, $GD = b'$, $EG = c$, $DF = c'$ $AA' = x$, $BB' = y$ and $CC' = z$. Evidently $A'C' = AC = \frac{1}{2}b'$

[1] (See Heath, Elementary Trigonometry, page 135.)

and $AC' = A'C = \frac{1}{2}b$. Similarly for other lines joining midpoints. Hence by Theorem IV, $x^2 + z^2 = \dfrac{b^2 + b'^2}{2}$,

$$y^2 + z^2 = \frac{a^2 + a'^2}{2} \quad \text{and} \quad x^2 + y^2 = \frac{c^2 + c'^2}{2}.$$

Eliminating x^2 and y^2 we get $4z^2 = a^2 + a'^2 + b^2 + b'^2 - c'^2 - c^2$. Hence the well-known problem: *In any quadrilateral, the sum of the squares of the sides equals the sum of the squares of the diagonals plus four times the square of the median of the diagonals.*

Fuerbach's theorem, which states that *the in-circle and the three ex-circles of a triangle are all tangent to the nine-point circle*, has many proofs. Hobson, in the Plane Trigonometry, gives a proof which makes use of Stewart's Theorem, but the proof is not suitable for this paper.

In 1875 Chasles gave a generalization of a particular case of the theorem and in 1891 M. Clément Thiry gave some interesting applications.

Stewart's theorem together with the four theorems of this paper are so intimately related to the Pythagorean theorem that it would almost seem to be an absolute necessity that every teacher of geometry in the secondary schools should have a knowledge of these. This enrichment affords such a delightful background. The books which the writer has used for reference are *A Sequel to Euclid* by John Casey, *College Geometry* by Altshiller-Court, *The Circle and the Sphere* by J. L. Coolidge, *Modern Geometry* by C. V. Durell, and *Modern Geometry* by Godfrey and Siddons.

DO NOT FORGET THE NATIONAL COUNCIL MEETING AT CLEVELAND ON FEBRUARY 22D AND 23D, 1928. THE COMPLETE PROGRAM WILL APPEAR IN A LATER NUMBER OF THE TEACHER.

A DIFFERENT BEGINNING FOR PLANE GEOMETRY

By H. C. CHRISTOFFERSON

Professor of Mathematics, Miami University, Oxford, Ohio

At a meeting of the Southeastern New York State Teachers Association held at Teachers College, Columbia University, last year, Dr. David Eugene Smith made the statement that there had been nothing new in geometry as it is taught in our high schools since the time of Euclid, 2200 years ago. Of course there have been minor changes, little tricks and devices, and even an introductory inductive chapter in many books, but the real demonstrative geometry is essentially the same in organization and content as established by that ancient but ingenious master mathematician and logician.

The ideas presented in this paper were stimulated by this challenge but are neither entirely original with the writer nor entirely new to the readers. They suggest an organization in the beginning of geometry which is different from that of Euclid or any textbook yet published. They were developed by the joint thought and effort of a class of graduate students at Teachers College last year.

At present all geometries begin their demonstration work with congruent triangles. First there will come "two sides and the included angle," then "two angles and the included side," then after perhaps a few trite unreal originals, "the isosceles triangle theorem" and finally "congruence by three sides." It is with these four theorems that the present paper is concerned.

Let us briefly review the proofs of the last two theorems. You will recall that in proving that the angles opposite the equal sides of an isosceles triangle are equal, it is necessary to bisect the angle at the vertex. Some texts seem to overlook the difficulty here and merely state the process as "draw *CD* bisecting angle *C*." Others are more shrewd and say "assume that *CD* is the bisector of angle *C*, since every angle must have a bisector." In the proof of congruence by three sides the isosceles triangle theorem is used to prove certain angles equal and then congru-

ence follows because of two sides and the included angle. Now after you have shown congruence by three sides it is possible to draw a bisector of any angle and prove the angle bisected using the "three sides" congruence theorem.

The circle reasoning here is evident and well known. In order to get any rigor in the reasoning process it is necessary to postulate a link somewhere in the chain. For many years we have used here what has been termed a hypothetical construction, that is, we have used a postulate variously worded but often phrased "every angle has a bisector." This construction is puzzling to teachers as well as children.

The proof for congruence by three sides depends upon the isosceles triangle theorem, the proof for the isosceles triangle theorem depends upon drawing an angle bisector, and the proof for drawing an angle bisector depends upon congruence by three sides. Clearly this chain of reasoning goes round in a circle. In order to make a rigorous system of reasoning we must postulate something; we must break the circle. This has been done since the time of Euclid by postulating the possibility of drawing an angle bisector. This solution has been approved by the Mathematical Association of America, but thinking teachers have never been satisfied with it. They argue that similarly an angle has trisectors but that would not permit us to postulate their construction in proving a theorem depending upon trisection. There must be some square exactly equal to a given circle in area but we can not postulate its existence in proving some theorem which depends on it. To be rigorous and yet not camouflage anything we need to postulate not merely the possibility of constructing a bisector, but the entire construction. Or what would be more simple, we should postulate congruence by three sides and on the basis of this prove the angle bisector construction, and the isosceles triangle theorem.

The postulation of congruence by three sides is by far the simplest solution of the problem in circle reasoning just described. This long pet proof in beginning geometry is such a sacred tradition that it will be difficult to persuade teachers to postulate it when they can give what seems to be a proof by means of a hypothetical construction. Logically this suggested solution is as defensible as the historic one we have been using. Psychologically it is better because it is more simple and straight-

forward. Technically it is far superior because of the splendid possibilities which it opens up for a more rigorous and interesting beginning for demonstrative geometry.

Let us examine the possibilities of this new approach. At present many textbooks are beginning their course in demonstrative geometry by a chapter of intuitive work. That intuitive work includes the construction of an angle, the bisector of an angle and of a line segment, the perpendicular to a line, and a triangle, given certain facts about it. There is no attempt to prove these constructions in this introductory chapter. Its purpose is merely to get the students accustomed to geometric terms and language before attempting demonstration.

The postulation of congruence by three sides, which postulation we have just defended from a logical standpoint, makes possible the more rigorous treatment of these construction exercises. In fact they may be readily and very simply proved, thus making the introductory chapter a real introduction to demonstrative plane geometry.

Suppose now that on the first day the geometry class met they were asked to draw a triangle, then to make an exact copy of that triangle, using ruler and compasses. Experience has shown that practically the entire class will measure off the three sides and draw the second triangle with its three sides equal respectively to the three sides of the first. A few may use two sides and the included angle and fewer two angles and the included side. From this copying the concept of congruent triangles is readily developed and the equality of corresponding angles is easily seen. Congruence by three sides can thus be postulated. The construction of an angle equal to a given angle can now be easily proved. Similarly the bisection of an angle, the construction of a perpendicular at a given point on a line can be rigorously and easily proved through using the postulate of congruence by three sides. These constructions make possible a host of others and geometry is well and rigorously begun. It would be easy to extend this article to a whole chapter of a book here. Thinking teachers will prefer to do this for themselves, however, so further details are omitted.

The lack of rigor in the proofs of the other two congruence theorems has long been recognized, yet few textbook writers have had the courage to refuse to follow Euclid in this respect.

Hilbert exposed the fallacy of the superposition proof, and progressive mathematicians no longer attempt to postulate superposition so as to prove the two congruence theorems when they can as well postulate the congruence theorems themselves, with no loss of rigor, and in fact with considerable gain. A few textbooks and writers have advocated the postulation of these two congruence theorems, so this is not new although rarely practiced.

In conclusion, we have contended for the postulation of congruence by two sides and the included angle, and also by two angles and the included side. Our major contention however is for postulation of congruence by three sides instead of postulation of the possibility of drawing an angle bisector. Such postulation is not less rigorous and far more direct and useful than the historic hypothetical construction. Furthermore such postulation makes possible an introductory chapter which is not merely intuitive, but rigorous, logical and yet simple demonstrative geometry.

The Mathematics Section of the Indiana State Teachers Association met in the Armory at Indianapolis, Indiana, on October 18th at 2 P.M. Mr. Prentice Edwards of Muncie presided. The following program was given: "Educational Tests—To Standardize or Not to Standardize"—W. D. Reeve, Teachers College. Discussion: W. E. Edington, Purdue University, "The New Course of Study in Mathematics"—Nelle Reed, Peru High School.

The Mathematics Section of the Connecticut State Teachers Association held its meeting in Room 26 of the Hopkins Street Building of the Hartford High School on Friday, October 26th, at 2 P.M. Miss Mary G. Worthley, of the Wm. H. Hall High School of West Hartford, presided. Over 300 teachers were in attendance.

THE VISUAL METHOD OF SOLVING ARITHMETIC PROBLEMS

By ARTHUR S. OTIS

Yonkers, N. Y.

The problem that is failed most often among the 20 reasoning problems in the Otis Arithmetic Reasoning Test[1] is this:

If a stick 20 inches long is cut into two pieces so that one piece is ⅔ as long as the other, how long will the longer piece be?

In order to best understand this article, you should stop just long enough now to work this problem. You might time yourself while you solve and check the solution. (You may use paper and pencil if you wish.) What answer did you get? The most common answer is 13⅓ inches, which of course is wrong. If you get that for an answer, you did not check your work. You might look at your watch and try again.

Now I claim that any pupil in the seventh grade, *if properly trained,* should be able to solve that problem in 30 seconds. How? This way: He may draw a few lines if necessary to help his reasoning, and think: "First I will draw a line to represent the longer piece." He draws any line, thus:

|————————————|

"Now the other piece is to be ⅔ as long, so I will divide my line into thirds (done roughly)

|————|————|————|

and add the other part, making it ⅔ as long as this first part."

|————|————|————|————|————|

"Now the whole line is 20 inches long, and it is divided into 5 equal parts, so that each part is ⅕ of 20 inches or 4 inches."

|————|————|————|————|————|
4 4 4 4 4

"The longer piece is 3×4 or 12 inches long. The other piece is 2×4 or 8 inches long. $8 + 4 = 12$ and $8 = ⅔$ of 12."

[1] Published by World Book Company, Yonkers-on-Hudson, N. Y.

What constitutes the usual source of error in solving this problem? First let us consider how the average person attacks it. There are two typical ways. This is the first: "One piece is to be $\frac{2}{3}$ as long as the other. But how long is the other? I don't see how I can find the length of either piece unless I know the length of the other. I guess I can't work it." This is the second: "One piece is to be $\frac{2}{3}$ as long as the other. $\frac{2}{3}$ of 20 is $13\frac{1}{3}$. The longer piece will be $13\frac{1}{3}$ inches long."

Wherein does the failure lie in this reasoning? In both cases it is the failure to realize the relationship between the parts and the whole. Even the pupil who visualizes the stick as a whole may feel helpless to find the point of division for lack of the simple

20 in.

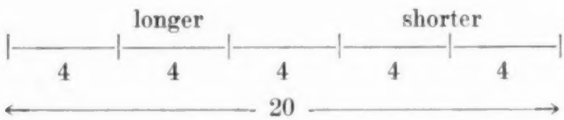
knowledge of where to begin the reasoning in order to find out. It is of course impossible to solve this problem without in some way taking account of the fact that the whole stick consists of 5 parts and that 2 of these constitute the shorter piece and 3 the longer piece. The trouble with most people is that they *do not know how to attack* the problem so as to find this number of parts (5) into which the whole stick must be thought of as divided. There are at least three ways in which this may be done. One method we might call abstract arithmetic. It is as follows:

(1) The shorter piece is $\frac{2}{3}$ of the longer.
(2) The longer piece is $\frac{3}{3}$ of itself.
(3) The two together are $\frac{5}{3}$ of the longer piece.
(4) $\frac{5}{3}$ of the longer piece $= 20$.
(5) $\frac{1}{3} = \frac{1}{5}$ of $20 = 4$.
(6) $\frac{3}{3} = 3 \times 4 = 12$.

The second method is just ordinary algebra as follows:

Let $x =$ the longer piece.
Then $\frac{2}{3}x =$ the shorter piece.
Then $x + \frac{2}{3}x =$ the whole.
That is, $\frac{5}{3}x = 20$.
$5x = 60$.
$x = 12$.

The third is the visual method we used above. The figure is here repeated:



Let us consider the difficulties of these three ways. First let us compare the method of abstract arithmetic with the visual method. Let us assume that the first step of each is of the same difficulty. Now what is there about writing down the fact that the shorter piece is $\frac{2}{3}$ of the longer that suggests that the next step is to write down that the longer piece is $\frac{3}{3}$ of itself? In the first place this second step sounds like a platitude. What point is there in saying that a thing equals $\frac{3}{3}$ of itself? And in the second place it is not suggested in the least by the first step.

Whereas, suppose we are using the visual method. How can we represent one line (B) as two thirds of another line (A) with-

out first finding $\frac{2}{3}$ of A by dividing it into 3 parts? You see the part of the visual solution that corresponds to step 2 of the abstract method is, we might say, automatic.

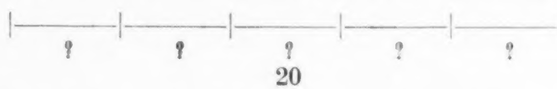
Now what is there about the first two steps of the abstract method that suggests the third? It is very difficult indeed for the pupil to get out of his mind the idea that the length of one piece must be gotten from the other. There is little or nothing to suggest to a pupil that he think of the two pieces together as a number of thirds of either one of them. Whereas, with the visual method, when the pupil has drawn the figure shown above, the fact that there are 5 equal parts in the stick as a whole *stares the pupil in the face!*

Another difficulty with step (3) of the abstract method is that the pupil has to add fractions, whereas by the visual method he merely adds short lines. A short line, when thought of as such, is just as much a unity as a long line. Indeed the pupil does not need even to add. He can count!

In each method of course the pupil must realize (4) that the whole stick equals 20 inches, and that he must next find the length of one of the parts. These steps are perhaps fairly evident in each method. But look at step (5) of the abstract method. It says "$\frac{1}{3} = \frac{1}{5}$." How can $\frac{1}{3} = \frac{1}{5}$? That is just too much for most children. Of course it means that $\frac{1}{3}$ of the longer

piece equals $\frac{1}{5}$ of the whole. That is just a little more than can be kept straight in the minds of most pupils. And why confuse them? Look at the pupil's drawing by the visual method. There is nothing but a line 20 units long divided into 5 equal parts. How long is one of them? Why $\frac{1}{5}$ of 20, of course! No thirds about it. There is where the secret of the visual method lies. While you are finding the length of one of the 5 equal parts of the whole, *you can forget thirds!*

Trying to keep several things in mind at once (thirds of one thing and fifths of another) leads to confusion and failure, whereas having only one thing to think of at a time (what is $\frac{1}{5}$ of 20?) makes for clear thinking and success. And of course



having found that each of the 5 equal parts of the whole is 4, it is quite obvious that the longer is 3×4 or 12 inches long

and the shorter is 2×4 or 8 inches long, and that (checking) $12 + 8 = 20$ and $8 = \frac{2}{3}$ of 12.

The method of abstract arithmetic therefore consists of a series of abstract statements. The second statement not only is not suggested by the first but is itself platitudinous. The third is not suggested by the second (thus making the start very difficult). And the fifth contains an apparent contradiction that is very difficult to comprehend.

The visual method calls for a drawing to illustrate the given statement that one piece of a stick is $\frac{2}{3}$ of the other. The one thing the pupil needs to be taught is that it is not necessary to begin by drawing the whole line. That is, we may begin by drawing either of the unknown parts. A line of any length will represent either part.

If the problem had stated that one piece of the stick was $1\frac{1}{2}$ times the other, then we would naturally begin with the shorter and add on another piece $1\frac{1}{2}$ times as long. But since the problem stated that one piece was $\frac{2}{3}$ as long as the other, we naturally begin with the longer one so we can add on another piece $\frac{2}{3}$ as long.

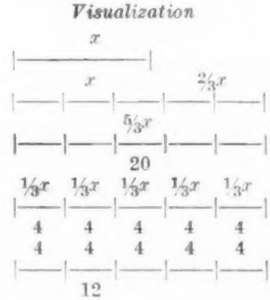
Hence it is seen that by the visual method the pupil has merely to know that he should make a drawing, and the making of it is almost self-directive and, when made, the deductions are almost automatic. All confusing complications are avoided. The pupil has but one thing to think of at a time. And the whole situation is completely before the eyes of the pupil as he does his reasoning.

The algebraic method is perhaps easier than the abstract arithmetic method after one has been taught to "let x equal the unknown called for," to express other unknowns in terms of the first, to form an equation to express known relationship, and to solve the equation by routine methods, using the axioms.

The main deficiency of the algebraic method is that it is all done in the dark, so to speak. As algebra it lacks the meaningfulness that the visual method carries. It is also mere abstract figuring. The equation is solved by rule and not by reason.

Of course one may carry visualization along with algebra; that is, one may accompany his algebraic solution with visual imagery or with graphic representation, thus:

Algebra	*Visualization*
1. Let $x =$ the longer piece.	1.
2. Then $\frac{2}{3}x =$ the shorter piece. 3. Then $x + \frac{2}{3}x =$ the whole.	2.
4. That is, $\frac{5}{3}x = 20$.	4.
5. $\frac{1}{3}x = 4$.	5.
6. $x = 12$.	6.



But note that the steps of visualization were themselves sufficient to effect a solution. So why bother with algebra? The point is that if an algebraic solution is accompanied by the sort of graphic representation that gives it meaning, the graphic representation itself constitutes a method of solution. Therefore, for a problem of this kind, algebra is either blind or needless. The visual method has a further advantage over algebra. It can be used mentally more easily. That is, with equal practice it is easier to visualize a line or two with cross marks than a whole set of algebraic equations.

There are, of course, many other kinds of problems that can

be solved visually in this same way. Here are some examples:

1. If alcohol and water are mixed in the ratio of 4 to 3, how much alcohol will be needed to make 21 qt. of a mixture?

2. If a set of drawings are to be engraved $\frac{4}{7}$ of their actual size, how long must a drawing be to come down to 8 in. when reduced?

3. What sum of money placed at interest at 5 percent a year for 4 years will amount to $300?

4. Divide the 32-in. inside vertical measurement of a bookcase by two 1-inch shelves so that each shelf space is 1 in. higher than the one above it.

5. Divide $25 between two boys so that one will have $6 more than the other.

6. Divide $27 among three boys so that *A* will have half as much as *B* and *B* will have a third as much as *C*.

7. What must a merchant charge for an overcoat that cost him $21 so he will gain 30 percent of the selling price?

8. A bottle and cork cost 22 cents. The bottle costs 20 cents more than the cork. What does the cork cost?

9. A turkey weighs 10 lb. and half his own weight. How much does he weigh?

10. A man paid $4 to get into a gambling place. While in he doubled his money, then he paid $4 to get out. He then had $10. What did he start with?

11. How deep a lot 50 ft. wide adjacent to a lot 100 ft. deep and 75 ft. wide will make the average depth of the two lots 125 ft.?

12. How much alcohol should be added to a 10-qt. mixture that is 50 percent alcohol to make it a mixture that is 75 percent alcohol?

13. How much mixture that is 75 percent alcohol should be added to 10 qt. of a mixture that is 50 percent alcohol to make a mixture that is 60 percent alcohol?

14. What value of goods (cost price) must be sold at a margin of 50 percent to make up for $5,000 worth (cost price) sold at a margin of 20 percent to make the average margin 30 percent?

15. What sum at 10 percent interest with $700 at 7 percent will yield 8 percent on the whole?

16. What amount of chocolate at 150° must be added to 2 cups at 90° to bring the mixture up to 110°?

17. I've worked 50 weeks at \$60 a week. How many weeks must I work at \$100 a week to bring my average up to \$75 a week? Solve mentally.

18. How much water must be distilled from a mixture that is 60 percent syrup to make it 80 percent syrup?

19. How much mass moving at 50 ft. a second will be required to strike a mass of 100 lb. at rest so that the whole mass will move at 20 ft. a second? (Mass $\times$ vel. = mass $\times$ vel.)

20. What sum at 10 percent plus \$500 at 5 percent plus \$700 at 7 percent will yield an average interest of 8 percent?

Nomination for New Members of the Board of Directors

All members of the National Council of Teachers of Mathematics are invited and urged to send to the Secretary-Treasurer their suggestions of nominees for the offices of Second Vice-President and three members of the Board of Directors to be elected at the Annual Meeting to be held in February, 1929. Please use the blank form below:

Second Vice-President: Nominee ________________

Three Members of Board: Nominees ________________

____

____

Mail this list of nominees to the Secretary-Treasurer, J. A. Foberg, State Teachers College, California, Penna.

The program of the Mathematics Section of the Association of Teachers in the Middle States and Maryland at the Atlantic City High School on December 1st is given below:

The Mathematical Prerequisites for Freshmen College Work.
Professor A. R. Congdon, University of Nebraska.

Round Table Discussion—New Methods in Teaching Geometry.

NEWS NOTES

The Third Annual Conference of Teachers of Mathematics in connection with the University of Iowa Extension Division (College of Education and Department of Mathematics Cooperating) was held at Iowa City on October 12th and 13th. The programs for the various sessions are as follows:

FRIDAY MORNING, OCTOBER 12
North Room, Old Capitol
H. L. RIETZ, *presiding*
Professor of Mathematics

9:30 A.M. Address: A Method of Dealing with Individual Differences without Classification
ELSIE PARKER JOHNSON
Oak Park High School, Oak Park, Illinois

10:00 A.M. Address: Some Points concerning the Selection of a Text in Plane Geometry
L. E. MENSENKAMP
Freeport High School, Freeport Illinois

10:30 A.M. Roots of Real and Complex Numbers
J. F. REILLY
Associate Professor of Mathematics
Discussion

FRIDAY AFTERNOON, OCTOBER 12
North Room, Old Capitol
R. P. BAKER, *presiding*
Associate Professor of Mathematics

1:30 P.M. Address: Present Trends in the Teaching of Elementary Mathematics
U. G. MITCHELL
Professor of Mathematics, University of Kansas

2:00 P.M. Address: Geometry Tests and Drills
RUTH LANE
University High School
Discussion

3:00 P.M. Symposium

FRIDAY EVENING, OCTOBER 12
Iowa Memorial Union

6:00 P.M. Conference dinner, followed by a talk by Mr. U. G. Mitchell on "Poetical Geography with Rules of Arithmetic in Rhyme" and by an informal symposium
ROSCOE WOODS, *toastmaster*
Assistant Professor of Mathematics

SATURDAY MORNING, OCTOBER 13
North Room, Old Capitol
E. W. CHITTENDEN, *presiding*
Professor of Mathematics

9:30 A.M. Address: Classification according to Ability in First Year Algebra
MR. L. E. MENSENKAMP

10:00 A.M. Address: Adapting the More Difficult Parts of Geometry to High School Pupils
MRS. JOHNSON

10:30 A.M. Address: A Different Point of View in the Teaching of Elementary Geometry
MR. MITCHELL
Discussion

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